The Finite Field Kakeya Problem

Aart Blokhuis and Francesco Mazzocca

Abstract

A Besicovitch set in $AG(n,q)$ is a set of points containing a line in every direction. The Kakeya problem is to determine the minimal size of such a set. We solve the Kakeya problem in the plane, and substantially improve the known bounds for $n > 4$.

1 Introduction

We denote by π_q the projective plane $PG(2,q)$ over the Galois field $GF(q)$ with q elements, $q > 2$ a prime power.

Let ℓ be a line in π_q and, for every point P on ℓ, let ℓ_P be a line on P other than ℓ. The set

$$K = (\bigcup_{P \in \ell} \ell_P) \setminus \ell \tag{1}$$

is called a *Kakeya set*, or a *minimal Besicovitch set.* The *finite plane Kakeya problem* asks for the smallest size $k(q)$ of a Kakeya set; it is the two-dimensional version of the *finite field Kakeya problem* posed by *T. Wolff* in his influential paper [11] of 1996.

In the following, unless explicitly mentioned otherwise, we will use the same notation of (1) for the lines defining a Kakeya set K.

Let Ω be a set of $q+2$ points in π_q. A point $P \in \Omega$ is said to be an *internal nucleus* of Ω if every line through P meets Ω in exactly one other point. Internal nuclei of $(q+2)-$sets were first considered by *A.Bichara* and *G.Korchmáros* in [1]; here they proved the following result.

Proposition 1 (1982) *Let q be an odd prime-power. Every set of $q+2$ points in π_q has at most two internal nuclei.*

The $q+2$ lines defining a Kakeya set in π_q can be viewed as a set of $q+2$ points with an internal nucleus in the dual plane π_q^*. More precisely, if K is a Kakeya set in π_q , the lines ℓ and ℓ_P, $P \in \ell$, give rise in π_q^* to a set $\Omega(K)$ of $q+2$ points with ℓ as an internal nucleus. Vice versa, every set of $q+2$ points with an internal nucleus in π_q defines in an obvious way a Kakeya set in π_q^*. Thanks to this duality, the finite plane Kakeya problem is equivalent to ask for *the smallest number* $k^*(q)$ *of lines in* π_q *meeting a set of* $q+2$ *points with an internal nucleus*; to be precise, we have

$$k^*(q) = 1 + q + k(q)\,.$$

2 Old and New Results in the Plane

Let us start by recalling that the first author and *A.A.Bruen* studied in [2] the smallest number of lines intersecting a set of $q+2$ points in π_q; here no assumption on the existence of internal nuclei is made. Nevertheless the dual of the theorem 1.3 of [2] contains the following result as a special case.

Proposition 2 (1989) *If* $q \geq 7$ *is odd, then*

$$|K| \geq \frac{q(q+1)}{2} + \frac{q+2}{3}\,,$$

for every Kakeya set K.

Example 1 Assume q is even and consider in π_q a dual hyperoval $\mathcal{H}$, i.e. a $(q+2)-$set of lines, no three of which are concurrent. Fix a line $\ell \in \mathcal{H}$ and, for every point $P \in \ell$, let ℓ_P the line of $\mathcal{H}$ on P other than ℓ. Then the Kakeya set

$$K(\mathcal{H}, \ell) = (\bigcup_{P \in \ell} \ell_P) \setminus \ell$$

is said to be associated to $\mathcal{H}$ and ℓ and it is of size

$$|K(\mathcal{H}, \ell)| = \frac{q(q+1)}{2}\,.$$

□

Example 2 Assume q is odd and consider in π_q a dual oval $\mathcal{O}$, i.e. a $(q+1)-$set of lines, no three concurrent. Let ℓ be a fixed line in $\mathcal{O}$. Every point P on ℓ, but one, belongs to a second line $\ell_P \in \mathcal{O}$ other than ℓ. If A is this remaining point on ℓ, let ℓ_A be a(ny) line through it different from ℓ. Then the Kakeya set

$$K(\mathcal{O},\ell,\ell_A) = (\bigcup_{P\in\ell} \ell_P) \setminus \ell$$

is said to be associated to $\mathcal{H}$, ℓ and ℓ_A; moreover it is of size

$$|K(\mathcal{O},\ell,\ell_A)| = \frac{q(q+1)}{2} + \frac{q-1}{2}\,.$$

□

For any point A of a Kakeya set K, we denote by m_A the number of lines ℓ_P , $P \in \ell$, on A and we set

$$\sigma(K) = \sum_{A\in K} \frac{(m_A-1)(m_A-2)}{2}\,. \tag{2}$$

In [7], *X. W. C. Faber* described special cases of *Examples* 1 and 2 and, by a counting argument, proved the following result.

Proposition 3 (Incidence formula, 2006) *The size of a Kakeya set K is given by*

$$|K| = \frac{q(q+1)}{2} + \sigma(K)\,. \tag{3}$$

Since $\sigma(K) \geq 0$, for every Kakeya set K, a first consequence of (3) is that

$$|K| \geq \frac{q(q+1)}{2}\,. \tag{4}$$

Let us note that *T. Wolff* in [11] proved that $|K| \geq q^2/2$; in fact his method gives inequality (4). Equality in (4) is actually attained in *Example* 1 and it is easy to see that this happens only in this case. So, when q is even, our problem is quite simple: *every Kakeya set K in π_q, q even, satisfies inequality* (4) *and equality holds iff K is associated to a dual hyperoval and one of its lines.*

When q is odd the plane π_q contains no hyperovals and $\sigma(K) > 0$, for every Kakeya set K. In this case the Kakeya set closest to that of *Example* 1 is the set $K(\mathcal{O},\ell,\ell_A)$ described in *Example* 2. This is the reason for the following conjecture recently raised and studied by *X. W. C. Faber* in [7].

Conjecture 1 (2006) *If q is odd, then*

$$|K| \geq \frac{q(q+1)}{2} + \frac{q-1}{2}\,,$$

for every Kakeya set K.

We remark that the *Blokhuis-Bruen* inequality in *Proposition* 2 is not so far from that of the conjecture. Moreover in [7], *X.W.C.Faber* obtained the following two results; the second one is a slight improvement of *Proposition* 2.

Proposition 4 (Triple point lemma, 2006) *Let K be a Kakeya set in π_q, q odd. Then, for every point $P \in \ell$, except possibly one, there exists a point $A \in \ell_P$ with $m_A \geq 3$.*

Proposition 5 (2006) *If q is odd, then*

$$|K| \geq \frac{q(q+1)}{2} + \frac{q}{3}, \tag{5}$$

for every Kakeya set K.

The *triple point lemma* is the the main tool in the proof of *Proposition* 5 and it is worth to remark that it is just the dual of *Proposition* 1. Actually it was proved by the same argument of *Bichara* and *Korchmáros* : the celebrated *Segre's lemma of tangents,* that was the key ingredient in his famous characterization of the $q+1$ rational points of an irreducible conic in π_q with q odd ([9]).

Let Ω be a $(q+2)-$set in π_q with an internal nucleus and let ℓ_∞ a line through this nucleus. Then, in the affine plane $AG(2,q) = \pi_q \setminus \ell_\infty$, the point set $\Omega \setminus \ell_\infty$ can be arranged as the graph $\{(a, f(a)) : a \in GF(q)\}$ of a function f, f being either a permutation or a semipermutation (i.e. a function whose range has size $q-1$) of $GF(q)$. This graph has been recently introduced and studied by *J.Cooper* in [6] and the following improvement to the *Faber's* inequality (5) has been obtained.

Proposition 6 (2006) *If q is odd, then*

$$|K| \geq \frac{q(q+1)}{2} + \frac{5q}{14} - \frac{1}{14}, \tag{6}$$

for every Kakeya set K.

Finally, we can settle Faber's conjecture, also characterizing the unique example realizing it. Actually we have the following sharp result.

Proposition 7 *If q is odd, then*

$$|K| \geq \frac{q(q+1)}{2} + \frac{q-1}{2},$$

for every Kakeya set K. Equality holds if and only if K is of type $K(\mathcal{O}, \ell, \ell_A)$, as in Example 2.

The essential ingredients in the proof are the *Segre's* lemma of tangents and the *Jamison-Brouwer-Schrijver* bound on the size of blocking sets in desarguesian affine planes ([3],[8]).

3 Solution of Kakeya's problem in the plane

We will give the proof of Proposition 7. It is more convenient however to phrase it in its dual form.

Proposition 8 *Let Ω be a set of $q+2$ points in $PG(2,q)$, with an internal nucleus. Then the number of lines intersecting Ω is at least*

$$k^*(q) = \frac{(q+1)(q+2)}{2} + \frac{q-1}{2}.$$

Equality implies that Ω consists of the points of an irreducible conic together with an external point.

Proof: Let a_i be the number of lines in AG(2,q) intersecting Ω in i points. Then:

$$\begin{cases} \sum a_i &= q^2+q+1 \\ \sum i\, a_i &= (q+2)(q+1) \\ \sum \binom{i}{2} a_i &= (q+2)(q+1)/2 \end{cases}$$

The first equation counts the total number of lines in the affine plane. In the second we count incident point-line pairs (P,ℓ), where P is a point of Ω. Finally in the third we count ordered triples (P,Q,ℓ), where P and Q are different points from Ω (and ℓ the unique line joining them). It follows that

$$a_0 + a_3 + 3a_4 + \ldots + \binom{q}{2} a_{q+1} = (q^2-q)/2.$$

Also, for later use we note that:

$$a_1 = 3a_3 + 8a_4 + \ldots = \sum_{n>2} (n^2-2n)a_n.$$

We aim for the situation where Ω is a conic together with an external point. In that case $a_1 = (q-1) + (q-1)/2$, $a_2 = (q^2+5)/2$, $a_3 = (q-1)/2$ and $a_0 = (q-1)^2/2$ (and the number of intersecting lines is $(q^2+4q+1)/2$).

Let the number of intersecting lines be $(q+2)(q+1)/2+f$ for some f, so that $a_0=(q^2-q)/2-f$. This gives us for f the equation

$$a_3+3a_4+\ldots+\binom{q}{2}a_{q+1}=f,$$

and we would like to show that $f\geq(q-1)/2$.

We know from *Bichara-Korchmáros* result (Prop.1), that there are at most 2 internal nuclei (in the example exactly 2) and by assumption there is at least one. Every other point is therefore on at least one tangent, and hence also on at least one (≥ 3)-secant. In particular $f\geq q/3$, with equality if every other point is on exactly one tangent and one three-secant (this does happen if $q=3$).

Every point, with the exception of the internal nucleus (nuclei), is on an odd intersector. So the odd intersectors form a blocking set of the dual affine plane if there is just one nucleus (this should maybe be called a dual blocking set, but we will use this term with a different meaning later). In this case:

$$a_1+a_3+a_5+\ldots\geq 2q-1,$$

and therefore

$$4a_3+8a_4+15a_5+\ldots\geq 2q-1,$$

and hence $f\geq(2q-1)/4$, more than we want.
From now on we assume that there are two internal nuclei, N_1 and N_2. Adding a random line on one of the internal nuclei, but not containing the other one, we again get a blocking set of the dual affine plane, and we obtain

$$4a_3+8a_4+15a_5+\ldots\geq 2q-2,$$

and hence $f\geq(2q-2)/4$ with equality if $a_k=0$ for $k>3$. So we have proved our lower bound, and we proceed to characterize the case of equality.
If $f=(q-1)/2$ then we have $(q-1)/2$ three-secants, and $3(q-1)/2$ tangents. Now if a point Q, is on exactly one tangent, and this happens often, then also on a unique three-secant, and we will show, that their intersection points with ℓ are related: if one is $(1:\lambda)$ the other is $(1:-\lambda)$), where coordinates are chosen such that $N_1=(1:0)$ and $N_2=(0:1)$.

Consider a three-secant containing two points on a unique tangent. Then these two tangents intersect in a point on the line joining the two internal nuclei (ℓ). This is true in the example and follows from

a *Segre*-type computation: if the three secant intersects the line ℓ in $(1:\lambda:0)$ then the unique tangents go through $(1:-\lambda:0)$), where the coordinates are set up in such a way that the two internal nuclei are $(1:0:0)$ and $(0:1:0)$.
We will use *Segre*-type computations a lot in the sequel. The general setup is the following. Consider three points $E_1=(1:0:0)$, $E_2=(0:1:0)$, $E_3=(0:0:1)$. Let X be any set of points such that no point of X is on one of the coordinate lines E_iE_j. For $x=(x_1:x_2:x_3)$ write down the triple $x'=(x'_1,x'_2,x'_3):=(x_2/x_1,x_3/x_2,x_1/x_3)$. It is clear from the definition that $\prod_{x\in X}x'_1x'_2x'_3=1$. On the other hand, it is sometimes possible, because of geometric properties of X to say something about $p_i=\prod_{x\in X}x'_i$. Applying this together with $p_1p_2p_3=1$ is called *Segre*'s lemma of tangents or a *Segre* computation. In our case the argument runs as follows. Let U be a point on a unique three-secant, further choose coordinates such that $U=(0:0:1)$, and some random fourth point equals $(1:1:1)$. Recall that $N_1=(1:0:0)$ and $N_2=(0:1:0)$. Let the three-secant through U intersect ℓ in $(1:\lambda:0)$ and let the unique tangent intersect ℓ in $(1:\mu:0)$. The remaining $q-1$ points of Ω (other than N_1, N_2 and U) have (homogeneous) coordinates $(a_i:b_i:c_i)$ with $a_ib_ic_i\neq 0$. We associate to such a point the triple $(b_i/a_i,c_i/b_i,a_i/c_i)$. Taking the product of all the entries in all triples we clearly get 1, because that is the contribution of each triple. On the other hand we have $\prod_i c_i/b_i=-1$, because on each line through N_1 we have a unique point of Ω so we just have the product of all non-zero field elements. In the same way $\prod a_i/c_i=-1$ by considering lines through N_2. To compute $\prod b_i/a_i$ we consider the lines through $U=(0:0:1)$. The three secant gives the value $b_i/a_i=\lambda$ twice, but the value $b_i/a_i=\mu$ is absent. All other nonzero field elements occur exactly once in the product, so for this product we end up with $-\lambda/\mu$, so $(-1)(-1)(-\lambda/\mu)=1$ and we conclude that $\mu=-\lambda$.

We will show that, unless $q=3$, the three points of Ω on a three-secant cannot all be points with a unique tangent, by applying again a *Segre* computation.
Apart from the 2 internal nuclei our set has q points, and all of them are on at least one tangent. The total number of tangents is

$$3(q-1)/2=q+(q-3)/2$$

hence at least $(q+3)/2$ points are on exactly one tangent (and one three-secant). So we certainly find a three-secant with (at least) two unique-tangent points on it.
Let $N_1=(1:0:0)$ and $N_2=(0:1:0)$ (as before) be the internal

nuclei.
Let $U_1 = (0:0:1)$ and $U_2 = (1:1:1)$ be two one-tangent points on a common three-secant, and let $V = (a:b:1)$ be a one-tangent point not on the line U_1U_2, so a and b are nonzero, and $a \neq b$.
Note that in our example we have that N_1, N_2, U_1 and U_2 are on a conic, and the tangents at $U_{1,2}$ are also known. So the conic has to be: $-2x_1x_2 + x_2x_3 + x_3x_1 = 0$. So we should expect that $-2ab + a + b = 0$ for $V = (a:b:1)$.
The three-secant U_1U_2 meets N_1N_2 in $(1:1:0) = N_1 + N_2$, so the tangents at U_1 and U_2 meet in $N_1 - N_2 = (1:-1:0)$. Let the tangent at V pass through $(1:\lambda:0)$, then the three-secant on V passes through the point $(1:-\lambda:0)$.
First we consider the triangle U_1N_1V. The tangent at V_1 intersects U_1N_1 in $U_1 + ((\lambda a - b)/\lambda)N_1$, the three-line in $U_1 + ((\lambda a + b)/\lambda)N_1$. The tangent through U_1 intersects N_1V in $N_1 + (-1/(a+b))V$, the three-line in $N_1 + (1/(b-a))V$. On VU_1 there are no special 'missing' or 'extra' points. *Segre* gives:

$$(a+b)(\lambda a + b) = (b-a)(\lambda a - b).$$

And we get the important fact $\lambda = -b^2/a^2$.
Next we consider the triangle $N_1U_2U_1$. Let the third point of Ω on U_1U_2 be $U_2 + \mu U_1$. On N_1U_2 we 'miss' the point $(-1:1:1) = N_1 + (-1/2)U_2$. On U_2U_1 we 'miss' the point $U_2 + \mu U_1$, and finally on U_1N_1 the point $(2:0:1) = U_1 + 2N_1$. Here we used that since the three-line on U_1 goes through $(1:1:0)$, the tangent passes through $(1:-1:0)$. It follows from the *Segre* product that $\mu = 1$.
We now turn to the triangle $U_1U_2V_1$. On U_1U_2 we find the 'extra' point, the intersection with the three line through V:

$$U_1 + \frac{(b+a\lambda)/(1+\lambda)}{1-(b+a\lambda)/(1+\lambda)}U_2.$$

and 'missing' points $U_1 + U_2$ (the third point of Ω on U_1U_2) and the intersection of the tangent through V with U_1U_2:

$$U_1 + \frac{(b-a\lambda)/(1-\lambda)}{1-(b-a\lambda)/(1-\lambda)}U_2.$$

This is of course just the expression for the three-secant with $-\lambda$ instead of λ. On U_2V and VU_1 we find 'missing' coordinates $-2/(a+b)$ and $-1 + (a+b)/2$. The *Segre* computation gives us

$$(a+b)(b+a\lambda)(1-b+(a-1)\lambda) = (a+b-2)(b-a\lambda)(1-b-(a-1)\lambda).$$

This we may rewrite as

$$a(a-1)\lambda^2 + (a+b-1)(a-b)\lambda + b(1-b) = 0.$$

Now substitute $\lambda = -b^2/a^2$, multiply by a^3 and divide by b. We get:

$$(a-b)(a+b)(2ab-a-b) = 0.$$

We already remarked that $a \neq b$, but also $a \neq -b$ because otherwise V would be on the tangent through U_1. Hence $2ab - a - b = 0$ and V is a point on the conic we are aiming for. A direct computation shows that also the tangent is 'right' and that the three-secant through V passes through the 'special point' $(1:1:2) = U_1 + U_2$.
Some counting to end the story. Let there be k points on a unique tangent. This means that our special point $U_1 + U_2$ is on at least $k/2$ three-secants, and hence on at least $k/2$ tangents. What is left in Ω (apart from the internal nuclei, the special point and the unique tangent points) is a set of $q-1-k$ points on at least 2 tangents, and a set of at most $3(q-1)/2 - k - k/2$ tangents. So

$$3(q-1)/2 - k - k/2 \geq 2(q-1-k).$$

This means $k \geq q-1$, so all other points are on the conic, and we finished the proof.

4 Applications to Dual Blocking Sets

A blocking set B in $\pi_q = PG(2,q)$ is a point set meeting every line and containing none.

Definition 1 *A dual blocking set S in π_q is a point set meeting every blocking set and containing no lines.*

Example 3 A Kakeya set $K = (\bigcup_{P\in\ell} \ell_P) \setminus \ell$ in π_q contains no lines. Moreover, for every blocking set B of π_q, a point P exists on $\ell \setminus B$ and so K meets B in a point of $\ell_P \setminus \ell$. It follows that K *is a dual blocking set.* □

Example 4 The complement $S = \pi_q \setminus (\ell \cup m)$ of the union of two distinct lines ℓ and m in π_q contains no lines. Moreover, no blocking set is contained in the union of two lines and so S meets every blocking set. It follows that S *is a dual blocking set.* □

Dual blocking sets were introduced by *P.Cameron, F.Mazzocca* and *R.Meshulam* in [4]; the first of the two main results of this paper is the following.

Proposition 9 (1988) *Let S be a dual blocking set in π_q. Then*

$$|S| \geq \frac{q(q+1)}{2}.$$

Equality holds if and only if either

(i) *S is the Kakeya set associated to a dual hyperoval and one of its lines; or*

(ii) *$q = 3$ and S is the complement of the union of two distinct lines.*

The argument in the proof of this proposition implicitly shows that every minimal (with respect to inclusion) dual blocking set in π_q is of one of types described in examples (3) and (4). For the sake of completeness we give an explicit proof of this result.

Proposition 10 *Let S be a minimal dual blocking set in π_q. Then one of the two following possibilities occur:*

(i) *$S = (\bigcup_{P \in \ell} \ell_P) \setminus \ell$ is a Kakeya set;*

(ii) *$S = \pi_q \setminus (\ell \cup m)$ is the complement of the union of two distinct lines ℓ and m.*

Proof: First of all we observe that there is a line ℓ disjoint from S, for if not, then, since S does not contain a line, S and its complement are blocking sets; a contradiction as S must meet every blocking set. Now we distinguish the following two cases.
Case 1. Assume that S is disjoint from exactly one line ℓ, and let P be a point of this line. If, for every line $m \neq \ell$ through P, there is a point $Q \neq P$ on m but not in S, then

$$B = (\ell \setminus \{P\}) \cup \left(\bigcup_{P \in m \neq \ell} m \right)$$

is a blocking set disjoint from S; a contradiction. Hence, for every point $P \in \ell$, there exists a line ℓ_P through P with $\ell_P \setminus \{P\} \subseteq S$. Then S contains the Kakeya set $K = (\bigcup_{P \in \ell} \ell_P) \setminus \ell$, which is a dual blocking set. From the minimality of S it follows that $S = K$.
Case 2. Assume that there are two lines ℓ and m disjoint from S. For any point $P \notin \ell \cup m$,let n be a line on P meeting $\ell \setminus m$ and $m \setminus \ell$ in the points L and M, respectively. Then $(\ell \cup m \cup \{P\}) \setminus \{L, M\}$ is a blocking set contained in $\ell \cup m \cup \{P\}$. It follows that P must belong to S and S is the complement of $\ell \cup m$. □

By *Propositions* 9 and 10 we can conclude that all the bounds previously shown for the size of a Kakeya set give, in the case that q is odd, corresponding new bounds for the size of a minimal dual blocking set, improving the result of *Proposition* 9. In fact, as a corollary of *Proposition* 7, we have the following sharp result.

Proposition 11 *Let S be a dual blocking set in π_q, q odd. Then*

$$|S| \geq \frac{q(q+1)}{2} + \frac{q-1}{2}$$

and equality holds if and only if S is a Kakeya set of type described in Example 2.

5 Old and new results in higher dimensions

In contrast to the plane case we only have bounds and conjectures for higher dimensions. In [11] it is shown that the number of points in a Kakeya set in AG(n, q) is at least $c \cdot q^{(n+2)/2}$, which is good for $n = 2$ but probably not for any larger n. The case $n = 3$ is the first open problem, but for $n = 4$ *T.Tao* has shown ([10]) that the exponent 3 can be improved to $3 + \frac{1}{16}$. In what follows we will show that for general n we get the lower bound $c \cdot q^{n-1}$, where $c = 1/(n-1)!$, so this improves the previous bounds when n is at least 5 and comes close to the conjectured $c_n q^n$. Unfortunately our ideas are for several reasons very unlikely to lead to improvements in the case of the 'real' Kakeya problem.
Very recently however, *Zeev Dvir* [5] has proved the finite field Kakeya problem, by showing that the number of points of a Kakeya set in AG(n, q) is at least $\binom{q+n-1}{n}$.

Since our result and proof are similar in nature but still slightly different, we will include it for historical reasons, and with the hope that an improved argument will give a bound equivalent or even slightly better than that of *Dvir.*

To improve the bound in higher dimensions we use a bound on the dimension of a certain geometric codes.

Consider the line-point incidence matrix of $PG(n, q)$. Number the points (so the columns): first the points in the hyperplane at infinity, then the points not in the Kakeya set, and finally the points in the Kakeya set. As usual we denote the number of points (and hyperplanes) in $PG(n, q)$ by $\theta_n = \theta_n(q) = (q^{n+1} - 1)/(q - 1)$.
Let the first θ_{n-1} rows be labeled by the lines defining the Kakeya

set, in the right order. The top consisting of the first θ_{n-1} rows of the incidence matrix now looks like this:

$$\mathbf{T} = (\mathbf{I}\,;\,\mathbf{O}\,;\,\mathbf{K})\,.$$

Here $\mathbf{I}$ is the identity matrix, and $\mathbf{K}$ is the θ_{n-1} by $|K|$ line-point incidence matrix of Kakeya-lines versus Kakeya-points Let $d = d_{n-1}$ be the dimension of C_{n-1}, the $GF(p)$-code (where $q = p^t$) spanned by the lines of $PG(n-1, q)$ (the hyperplane at infinity). Then there is a subset C of the points, of size $\theta_{n-1} - d_{n-1}$ that does not contain the support of a codeword (this is obvious: after normalization a generator matrix for this code has the form $(I\,;\,A)$ and every nonzero codeword has a nonzero coordinate in one of the first d_{n-1} positions, so no codeword has its support contained in the 'tail' of length $\theta_{n-1} - d_{n-1}$). It follows that the set of Kakeya points has at least this size: Consider the $\theta_{n-1} - d_{n-1}$ rows of $\mathbf{T}$ corresponding to the Kakeya lines having a direction in C. Suppose the corresponding rows of $\mathbf{K}$ are dependent (over $GF(p)$). Then this dependency would produce a codeword in the line-point code of $\mathrm{PG}(n, q)$ with support contained in the set C in the hyperplane at infinity. But such a word is already in the point-line code of this hyperplane. To see this, let C_n stand for the line code of $PG(n, q)$, and C_{n-1} for the line code of the hyperplane H. Clearly $C_n^{\perp}|_H \subseteq C_{n-1}^{\perp}$. We show that in fact equality holds, for let u be a word in $C_{n-1}^{\perp}$, and now take a point $P \notin H$ and form the cone with top P over u, but remove P. This defines in an obvious way a word $\tilde{u}$ in $C_n^{\perp}$ whose restriction to H is u.
So we find $|K| \geq \dim C_{n-1}^{\perp}$. The dimension of C_{n-1} is known, and equal to something complicated. For us the bound

$$|K| \geq \dim C_{n-1}^{\perp} \geq \binom{q+n-2}{n-1} \geq q^{n-1}/(n-1)!$$

suffices. In fact, if q is prime we have equality, if not we have a little improvement, but not an essential one.

ACKNOWLEDGMENTS This research was done when the first author was visiting the *Seconda Universitá degli Studi di Napoli* at Caserta. The authors wish to thank for their supports the research group GNSAGA of Italian *Istituto Nazionale di Alta Matematica* and the *Mathematics Department* of the *Seconda Università degli Studi di Napoli.*

References

[1] A. Bichara, G. Korchmáros: Note on $(q+2)-$Sets in a Galois Plane of Order q, *Ann. Discrete Math.*, **14** (1982), 117-121.

[2] A. Blokhuis, A.A. Bruen: The Minimal Number of Lines Intersected by a Set of $q+2$ Points, Blocking Sets, and Intersecting Circles, *J. Combin. Theory* Ser.A, **50** (1989), 308-315.

[3] A.E. Brouwer, A. Schrijver: The Blocking Number of an Affine Space, *J. Combin. Theory* Ser.A, **24** (1978), 251-253.

[4] P.J. Cameron, F. Mazzocca, R. Meshulam: Dual Blocking Sets in Projective and Affine Planes, *Geom.Dedicata*, **27** (1988), 203-207.

[5] Z. Dvir: On the Size of Kakeya Sets in Finite Fields. *J. of the AMS* (to appear), (2008).

[6] J. Cooper: Collinear Triple Hypergraphs and the Finite Plane Kakeya Problem, *Arxiv preprint* math.CO/0607734, (2006) - arxiv.org.

[7] X.W.C. Faber: On the Finite Field Kakeya Problem in Two Dimensions, *J. Number Theory*, **117** (2006), 471-481.

[8] R. Jamison: Covering Finite Fields with Cosets of Subspaces, *J. Combin. Theory* Ser.A, **22** (1977), 253-266.

[9] B. Segre: Ovals in Finite Projective Planes, *Canad.J.Math.*, **7** (1955), 414-416.

[10] T. Tao: A New Bound for Finite Field Besicovitch Sets in Four Dimensions, *Pacific J. Math*, **222** (2005), 337-364.

[11] T. Wolff: Recent Work Connected whith the Kakeya Problem, *Prospects in Matematics* (Princeton, NJ, 1996), Amer. Math. Soc., Providence, RI, (1999), 129-162.

Authors address:

A.Blokhuis
Eindhoven University of Technology
Department of Mathematics and Computing Science
P.O. Box 513
5600 MB Eindhoven, The Nederlands
e-mail: `a.blokhuis@tue.nl`

F.Mazzocca
Seconda Università degli Studi di Napoli
Dipartimento di Matematica
via Vivaldi 43,
I-81100 Caserta, Italy
e-mail: `francesco.mazzocca@unina2.it`