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DUAL BLOCKING SETS IN PROJECTIVE AND AFFINE PLANES

ABSTRACT. A dual blocking set is a set of points which meets every blocking set but contains no line. We establish a lower bound for the cardinality of such a set, and characterize sets meeting the bound, in projective and affine planes.

A *blocking set* for a family $\mathscr{F}$ of sets is a set which meets every member of $\mathscr{F}$ but contains none. Blocking sets have been studied intensively, especially in the case where $\mathscr{F}$ is the set of lines of a projective or affine plane (see, for example, [1]). Two of the motivating questions are: What is the minimum size of a blocking set? and What is the structure of blocking sets of minimal size?

A *dual blocking set* for $\mathscr{F}$ is a set which meets every blocking set for $\mathscr{F}$ but contains no member of $\mathscr{F}$. In the course of showing that a projective or affine plane is, in general, determined by its family of blocking sets, the first two authors showed in [2] that, for such planes, the smallest sets meeting every blocking set are the lines: in other words, a dual blocking set has larger cardinality than a line. The question 'How much larger?' was left open. We propose to answer that question here.

We require one further definition in order to state our results. A *line oval* in a projective plane of order n is a set of $n + 2$ lines, no three concurrent. A *line oval* in an affine plane of order n is a line oval in the corresponding projective plane, one of whose lines is the line at infinity: in other words, a set of $n + 1$ lines, no three concurrent and no two parallel. It is well known that line ovals exist only in planes of even order, and that any Desarguesian plane of even order contains them.

Projective planes of order 2, and affine planes of order 2 or 3, contain no blocking sets; so we exclude these.

THEOREM 1. *Let S be a dual blocking set in a projective plane of order $n \geqslant 3$. Then $|S| \leqslant \frac{1}{2}n(n+1)$. Equality holds if and only if either*

(a) *$n = 3$ and S is the complement of the union of two lines; or*
(b) *$S = \bigcup\mathscr{L}\setminus L$, where $\mathscr{L}$ is a line oval and $L \in \mathscr{L}$.*

THEOREM 2. *Let S be a dual blocking set in an affine plane of order $n \geqslant 4$. Then $|S| \leqslant \frac{1}{2}n(n-1)$. Equality holds if and only if $S = \bigcup\mathscr{L}\setminus L$, where $\mathscr{L}$ is a line oval and $L \in \mathscr{L}$.*

Before beginning the proof, we note one general result:

PROPOSITION. *If S is a dual blocking set for $\mathscr{F}$, and is minimal* (*with respect to inclusion*) *subject to this, then there is a member of $\mathscr{F}$ disjoint from S.*

Proof. If not, then S meets every member of $\mathscr{F}$ and every $\mathscr{F}$-blocking set,

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and hence meets every set meeting every member of $\mathscr{F}$. Thus, the complement of S is disjoint from some member of $\mathscr{F}$.

PROOF OF THEOREM 1. Clearly we may assume that S is minimal with respect to inclusion, among dual blocking sets. Then there is a line L satisfying $S \cap L = \emptyset$, by the proposition. We distinguish two cases:

Case 1. S is disjoint from exactly one line L. Let x be a point of L. If, for each line $M \neq L$ incident with x, there is a point $y \neq x$ on M but not in S, then the complement of $S \cup \{x\}$ is a blocking set disjoint from S; a contradiction. Hence, for every point $x \in L$, there is a line $M(x)$ on x with $|M(x) \cap S| = n$. Then

$$\begin{aligned} |S| &\geqslant \left| \bigcup_{x \in L} M(x) \cap S \right| \\ &\geqslant (n+1)n - \sum_{\substack{x, y \in L \\ x \neq y}} |M(x) \cap M(y) \cap S| \\ &= (n+1)n - \binom{n+1}{2} \\ &= \tfrac{1}{2}(n+1)n. \end{aligned}$$

Suppose that equality holds. The tightness of the first inequality shows that

$$S = \bigcup_{x \in L} M(x) \cap S = \left(\bigcup_{x \in L} M(x) \right) \backslash L.$$

The tightness of the second inequality shows that any point of S lies on just two lines $M(x) (x \in L)$, so that $\{L\} \cup \{M(x) : x \in L\}$ is a line oval, and the second alternative of the theorem holds.

Conversely, let $\mathscr{L}$ be a line oval, and $L \in \mathscr{L}$. For $x \in L$, let $M(x)$ be the line of $\mathscr{L}$, different from L, contaning x. If B is contained in the complement of $(\bigcup \mathscr{L}) \backslash L$, then either $L \subseteq B$, or there is a point $x \in L \backslash B$; in the latter case, $B \cap M(x) = \emptyset$. In either case, B is not a blocking set. Thus $(\bigcup \mathscr{L}) \backslash L$ is a dual blocking set. Clearly it is minimal.

Case 2. There are two lines L, M disjoint from S. For any point $x \notin L \cup M$, the set $L \cup M \cup \{x\}$ contains a blocking set – if N is a line on x meeting L and M in p and q respectively, where $p \neq q$, then $L \cup M \cup \{x\} \backslash \{p, q\}$ is a blocking set. So S contains each such x. Thus S is the complement of $L \cup M$, and

$$|S| = n(n-1) \geqslant \tfrac{1}{2}n(n+1).$$

since $n \geqslant 3$; equality holds only if $n = 3$.

Conversely, the complement of the union of two lines is a dual blocking set, since no blocking set is contained in the union of two lines; and it is minimal with respect to inclusion, by the above argument.

PROOF OF THEOREM 2. The argument is similar to that for Theorem 1, though more elaborate. Let S be a dual blocking set, minimal under inclusion. By the proposition, there is a line disjoint from S.

Case 1. S is disjoint from exactly one line L. As in Theorem 1, we deduce that for each $x \in L$, there exists a line $M(x) \neq L$ on x such that $M(x)\backslash\{x\} \subseteq S$. This yields

$$\begin{aligned} |S| &\geqslant \sum_{x \in L} |M(x)\backslash\{x\}| - \sum_{\substack{x, y L \\ x \neq y}} |M(x) \cap M(y)| \\ &\geqslant n(n-1) - \binom{n}{2} \\ &= \tfrac{1}{2}n(n-1). \end{aligned}$$

Equality implies both that

$$S = \bigcup_{x \in L} M(x)\backslash L,$$

and that $M(x) \cap M(y) \neq \emptyset$ but $M(x) \cap M(y) \cap M(z)) = \emptyset$ for all distinct $x, y, z \in L$, so that $\mathscr{L} = \{L\} \cup \{M(x) : x \in L\}$ is a line oval.

Case 2. S is disjoint from more than one line, say $L_1, \ldots, L_k (k \geqslant 2)$.

Subcase A. Three of the lines L_i, say L_1, L_2, L_3, are not all parallel. Then there is a blocking set contained in $L_1 \cup L_2 \cup L_3$, and hence disjoint from S; a contradiction. (If $L_1 \| L_2$, and $L_i \cap L_3 = \{x_i\}$ for $i = 1, 2$, then $L_1 \cup L_2 \cup L_3 \backslash \{x_1, x_2\}$ is a blocking set. If no two of the three lines are parallel, and $n \geqslant 5$, choose x_i lying on L_i but neither of the other two lines for $i = 1, 2, 3$, such that the line $x_i x_j$ is not parallel to L_k whenever $\{i, j, k\} = \{1, 2, 3\}$; then $L_1 \cup L_2 \cup L_3 \backslash \{x_1, x_2, x_3\}$ is a blocking set. In the case $n = 4$, the plane is isomorphic to AG(2, 4), and the claim may be verified directly.)

Subcase B. $L_1, \ldots, L_k$ are all parallel. Choose a point x not in S and lying on no line L_i. (This is possible since S contains none of the remaining lines parallel to L_1.) Then, if L is a line through x not parallel to L_1, the complement of $S \cup (L \cap \bigcup_{i=1}^{k} L_i)$ is a blocking set; a contradiction.

Subcase C. $k = 2$ and L_1, L_2 intersect. Let $U_1, \ldots, U_{n-1}$ be the further lines parallel to L_1, and $V_1, \ldots, V_{n-1}$ those parallel to L_2. Let x (resp. y) be the number of lines U_i (resp. V_i) which do *not* contain $n - 1$ points of S.

LEMMA. *If* $|U_t \cap S| \leqslant n - 2$, *then* $|U_i \cap S| \geqslant y$ *for all* $i \neq t$.

Proof. Let p be the point on U_t and L_2. For each j such that $|V_j \cap S| \leqslant n - 2$, let q_j be the point on V_j and L_1, and r_j the intersection of pq_j and U_i. We claim that $r_j \in S$. Indeed, if $r_j \notin S$, then the complement of $S \cup \{p, q_j\}$ would be a blocking set. Since the points r_j are distinct, $|U_i \cap S| \geqslant y$.

Without loss of generality, we may suppose that $y \geqslant x$. If $x = 1$ then $|U_i \cap S| = n - 1$ for, say, $1 \leqslant i \leqslant n - 2$, and $U_{n-1} \cap S \neq \emptyset$; so $|S| \geqslant (n-1)(n-2) + 1 > \frac{1}{2}n(n-1)$.

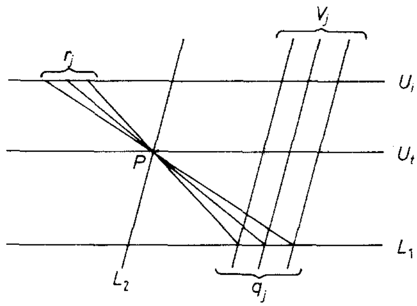

Fig. 1.

So we may suppose that $x \geqslant 2$. Now the lemma shows that $|U_i \cap S| \geqslant y$ for all i, and thus

(1) $$|S| \geqslant (n-1-x)(n-1) + xy.$$

On the other hand, clearly

(2) $$|S| \geqslant (n-1)^2 - xy,$$

since the lines U_i, V_j containing a point not in $S \cup L_1 \cup L_2$ satisfy $|U_i \cap S| < n-1$ and $|V_j \cap S| < n-1$; and therefore

(3) $$\begin{aligned} |S| &\geqslant \tfrac{1}{2}((n-1-x)(n-1) + (n-1)^2) \\ &= \tfrac{1}{2}(n-1)(2n-2-x). \end{aligned}$$

Hence, if $x \leqslant n-3$, then $|S| \geqslant \frac{1}{2}(n-1)(n+1) > \frac{1}{2}n(n-1)$. Otherwise $y \geqslant x \geqslant n-2$, and so, by (1), we have $|S| \geqslant n-1+(n-2)^2 > \binom{n}{2}$, proving the theorem.

REMARK. Equations (1)–(3) imply that, in Case 2, $|S| \gtrsim cn^2$ for some $c \geqslant \frac{1}{2}$. A slightly better bound is obtained as follows. From the proof of the lemma, if $x, y \geqslant 1$, then

$$|S| \geqslant (n-2)x$$

and

$$|S| \geqslant (n-2)y$$

and so

$$|S|^2 \geqslant (n-2)^2 xy \geqslant (n-2)^2((n-1)^2 - |S|).$$

Therefore

$$|S| \gtrsim \tfrac{1}{2}(\sqrt{5}-1)n^2.$$

REFERENCES

1. Bruen, A. A., 'Baer Subplanes and Blocking Sets', *Bull. Amer. Math. Soc.* **76** (1970), 342–344.
2. Cameron, P. J. and Mazzocca, F., 'Bijections which Preserve Blocking Sets', *Geom. Dedicata* **21** (1986), 219–229.

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