

Blocking Sets in $PG(r, q^n)$

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Abstract Let \mathcal{S} be a Desarguesian $(n - 1)$ -spread of a hyperplane Σ of $PG(rn, q)$. Let Ω and \bar{B} be, respectively, an $(n - 2)$ -dimensional subspace of an element of \mathcal{S} and a minimal blocking set of an $((r - 1)n + 1)$ -dimensional subspace of $PG(rn, q)$ skew to Ω . Denote by K the cone with vertex Ω and base \bar{B} , and consider the point set B defined by

$$B = (K \setminus \Sigma) \cup \{X \in \mathcal{S} : X \cap K \neq \emptyset\}$$

in the Barlotti–Cofman representation of $PG(r, q^n)$ in $PG(rn, q)$ associated to the $(n - 1)$ -spread \mathcal{S} . Generalizing the constructions of Mazzocca and Polverino (J Algebraic Combin, 24(1):61–81, 2006), under suitable assumptions on \bar{B} , we prove that B is a minimal blocking set in $PG(r, q^n)$. In this way, we achieve new classes of minimal blocking sets and we find new sizes of minimal blocking sets in finite projective spaces of non-prime order. In particular, for q a power of 3, we exhibit examples of r -dimensional minimal blocking sets of size $q^{n+2} + 1$ in $PG(r, q^n)$, $3 \leq r \leq 6$ and $n \geq 3$, and of size $q^4 + 1$ in $PG(r, q^2)$, $4 \leq r \leq 6$; actually, in the second case, these blocking sets turn out to be the union of q^3 Baer sublines through a point. Moreover, for q an even power of 3, we construct examples of minimal blocking sets of $PG(4, q)$ of size at least $q^2 + 2$. From these constructions, we also get maximal partial ovoids of the hermitian variety $H(4, q^2)$ of size $q^4 + 1$, for any q a power of 3.

Keywords Blocking set · Ovoid · Spread

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1 Introduction

Let $PG(r, q)$ denote the r -dimensional projective space associated with the $(r + 1)$ -dimensional vector space $GF(q)^{r+1}$ over the finite field $GF(q)$ with q elements. Here, always $q = p^h$, p prime, $h \geq 1$. Following [10], a *blocking set* in $PG(r, q)$, $r \geq 2$, is defined as a point set B intersecting every hyperplane and containing no line. A subspace S of $PG(r, q)$ meeting a blocking set B just in one point P is said to be *tangent* to B in P . Moreover, a point P of B is said to be *essential* if $B \setminus \{P\}$ is not a blocking set, that is, if a tangent hyperplane to B in P exists. When all points of B are essential, no proper subset of B is a blocking set and B is called *minimal*. It is straightforward to see that, if B is a minimal blocking set in $PG(r, q)$, then B is a blocking set in any projective subspace of $PG(r, q)$ containing it and the minimality is preserved. Finally, we say that a blocking set B is *h -dimensional* if the subspace generated by B has dimension h ; in particular, for $h = 2$, we say that B is *planar*.

The terminology on blocking sets is unfortunately not yet standard, so slight variations of the previous definitions can be found. For example, in [6], a blocking set in $PG(r, q)$ is defined as a *1-blocking set*. Now, we will survey some results useful in what follows.

Baer subplanes and unital in $PG(2, q^2)$, as well as ovoids in $PG(3, q)$, are examples of extremal minimal blocking sets, in the sense of the following two classical results.

Result 1 (Bruen [9] for $r = 2$; Beutelspacher [6] for $r > 2$) *The minimal possible size of a blocking set B in a finite projective space $PG(r, q)$, $r \geq 2$, is $q + \sqrt{q} + 1$ and the bound is attained if and only if q is a square and B is a Baer subplane.*

In the case $r > 2$, improved results have been obtained by Storme and Weiner [18].

Result 2 (Bruen and Thas [10]) *Let B be a minimal blocking set in $PG(r, q)$. Then we have the following results:*

- if $r = 2$, $|B| \leq q\sqrt{q} + 1$ and equality holds if and only if q is a square and B is a unital;
- if $r = 3$, $|B| \leq q^2 + 1$ and equality holds if and only if B is an ovoid;
- if $r \geq 4$, $|B| < \sqrt{q^{r+1}} + 1$.

An ovoid of $PG(3, q)$, $q > 2$, is a $(q^2 + 1)$ -cap. An ovoid of $PG(3, 2)$ is a complete 5-cap of $PG(3, 2)$. Every plane of $PG(3, q)$ intersects an ovoid in either one or $q + 1$ points. Furthermore, every point of an ovoid belongs to a unique plane intersecting this ovoid in one point. This shows that an ovoid is a minimal blocking set of $PG(3, q)$.

The notion of ovoid can be generalized to a non-singular quadric Q of $PG(r, q)$ and to a non-singular hermitian variety $H(r, q^2)$ of $PG(r, q^2)$: it is a point set of Q or $H(r, q^2)$ meeting exactly once every generator (i.e. maximal subspace) of Q or $H(r, q^2)$, respectively.

An ovoid of a non-singular parabolic quadric $Q(2r, q)$ in $PG(2r, q)$ contains exactly $q^r + 1$ points and it is known that it can only exist if $r = 2, 3$ (Gunawardena and Moorhouse [11] for q odd, Thas [21] for q even). The three-dimensional elliptic quadrics of $Q(4, q)$ are the classical ovoids of $Q(4, q)$. Furthermore, $Q(6, q)$ has ovoids if $q = 3^h$, $h \geq 1$, but $Q(6, q)$ has no ovoids if $q > 3$ prime [4]. Every ovoid of $Q(2r, q)$, $r = 2, 3$, is a minimal blocking set in $PG(2r, q)$, and meets every hyperplane in $1 \pmod p$ points; p the characteristic

of $GF(q)$ (Ball [3], Ball et al. [4]). So far, for $n = 2, 3$, ovoids of $Q(2n, q)$ are the largest known examples of minimal blocking sets in $PG(2n, q)$.

An ovoid of $H(r, q^2)$ has size $q^{r+1} + 1$ if r is even and $q^r + 1$ if r is odd. In [21], it is proved that $H(2r, q^2)$, with $r \geq 2$, has no ovoid. An ovoid of $H(3, q^2)$ is said to be *locally hermitian* with respect to one of its points R if it is the union of q^2 Baer sublines through R . A *partial ovoid* of $H(r, q^2)$ is a set of points of $H(r, q^2)$ meeting every generator of $H(r, q^2)$ in at most one point and it is said to be *maximal* if it is not a subset of a larger partial ovoid.

In this paper, by using the Barlotti–Cofman representation of a projective space of non-prime order [5], we generalize the *cone constructions* described in [15, 19] and we achieve new classes and new sizes of minimal blocking sets in $PG(r, q^n)$. Choosing ovoids of $PG(3, q)$, $Q(4, q)$ and $Q(6, q)$ as base of our cones, we also obtain some *large* blocking sets (see Theorems 2, 4, 7–10). In particular, for q an even power of 3, we obtain in $PG(4, q)$ minimal blocking sets of size $q^2 + 1$ different from ovoids of parabolic quadrics, and minimal blocking sets of size at least $q^2 + 2$. Finally, for q a power of 3, we construct maximal partial ovoids of $H(r, q^2)$, $4 \leq r \leq 6$, of size $q^4 + 1$, which are the union of Baer sublines through a point.

2 The constructions

An $(n - 1)$ -spread of a projective space $PG(m - 1, q)$ is a family \mathcal{S} of mutually disjoint subspaces of dimension $n - 1$ such that each point of $PG(m - 1, q)$ belongs to an element of \mathcal{S} . It has been proved by Segre [17] that $(n - 1)$ -spreads of $PG(m - 1, q)$ exist if and only if $m = rn$. If $r > 2$, an $(n - 1)$ -spread \mathcal{S} of $PG(rn - 1, q)$ is said to be *normal* if it induces an $(n - 1)$ -spread in any subspace generated by two of its elements, i.e. if $A, B \in \mathcal{S}$, then any element of \mathcal{S} is either disjoint from $T = \langle A, B \rangle$ or contained in T . We say that an $(n - 1)$ -spread \mathcal{S} of $PG(rn - 1, q)$ is *Desarguesian* if either $r > 2$ and \mathcal{S} is normal, or $r = 2$ and the associated translation plane obtained by André–Bruck and Bose representation (see [1, 7, 8]) is Desarguesian.

Let \mathcal{S} be a Desarguesian $(n - 1)$ -spread of $\Sigma = PG(rn - 1, q)$ and define an \mathcal{S} -subspace of Σ as a subspace X of Σ whose points are partitioned by elements of \mathcal{S} . If X is an \mathcal{S} -subspace different from a spread element, then \mathcal{S} induces on it a Desarguesian $(n - 1)$ -spread and its dimension is of type $tn - 1$, with $2 \leq t \leq r$.

The $(n - 1)$ -spread \mathcal{S} determines a representation $PG(\mathcal{S})$ of $PG(r - 1, q^n)$ in $PG(rn - 1, q)$ where points and t -subspaces of $PG(r - 1, q^n)$, $0 < t < r - 1$, correspond to elements of \mathcal{S} and \mathcal{S} -subspaces of dimension $(t + 1)n - 1$ of Σ , respectively.

Remark 1 Every hyperplane H of Σ contains exactly one \mathcal{S} -subspace of dimension $(r - 1)n - 1$. This \mathcal{S} -subspace is the union of all spread elements contained in H .

Now, embed Σ as a hyperplane in $\Sigma' = PG(rn, q)$ and define a point-line geometry $\Pi_r = \Pi_r(\Sigma', \Sigma, \mathcal{S})$ in the following way:

- the points of Π_r are the points of $\Sigma' \setminus \Sigma$ and the elements of \mathcal{S} ;
- the lines of Π_r are the n -subspaces of Σ' intersecting Σ in an element of \mathcal{S} and the lines of $PG(\mathcal{S})$;
- the point-line incidences are inherited from Σ and Σ' .

The incidence structure Π_r is isomorphic to the projective space $PG(r, q^n)$ and we say that Π_r is the *Barlotti–Cofman representation* of $PG(r, q^n)$ (see [5]). The points of Π_r in $\Sigma' \setminus \Sigma$ will be called *affine*. Note that a t -subspace of Π_r is either a tn -subspace of Σ' meeting

Σ in an \mathcal{S} -subspace of dimension $tn - 1$, or a t -subspace of $PG(\mathcal{S})$, i.e. an \mathcal{S} -subspace of Σ of dimension $(t + 1)n - 1$. In particular, $PG(\mathcal{S})$ is a hyperplane of Π_r .

Let Y be a fixed element of \mathcal{S} and let $\Omega = \Omega_{n-2}$ be a hyperplane of Y . Let $\Gamma' = \Gamma'_{(r-1)n+1}$ be an $((r - 1)n + 1)$ -subspace of Σ' disjoint from Ω . Also, denote by $\Gamma = \Gamma_{(r-1)n}$ the $(r - 1)n$ -subspace intersection of Γ' and Σ , and by T the intersection point of Γ and Y .

Let \bar{B} be a blocking set of Γ' such that $\bar{B} \cap \Gamma = \{Q\}$, Q a point, with the following property:

(α) $\ell \setminus \{T\} \not\subset \bar{B}$, for every line ℓ of Γ' through T .

Denote by K the cone with vertex Ω and base \bar{B} , i.e.

$$K = \bigcup_{\bar{P} \in \bar{B}} \langle \bar{P}, \Omega \rangle$$

and note that, since $\Gamma' \cap \Omega = \emptyset$, $\langle \bar{P}, \Omega \rangle \cap \langle \bar{P}', \Omega \rangle = \Omega$, for any distinct points $\bar{P}, \bar{P}' \in \bar{B}$. Let B be the subset of Π_r defined by

$$B = (K \setminus \Sigma) \cup \{X \in \mathcal{S} : X \cap K \neq \emptyset\}$$

and note that if $Q \in Y$ (i.e. $T = Q$), then $|B| = q^{n-1}(|\bar{B}| - 1) + 1$ and $B \cap PG(\mathcal{S}) = \{Y\}$; while if $Q \notin Y$, then $|B| = q^{n-1}|\bar{B}| + 1$ and $|B \cap PG(\mathcal{S})| = q^{n-1} + 1$.

Proposition 1 *B is a blocking set of the projective space Π_r .*

Proof Let $S_{(r-1)n}$ be an $(r - 1)n$ -dimensional subspace of Σ' not contained in Σ and skew to Ω . Then $\dim \langle S_{(r-1)n}, \Omega \rangle = rn - 1$ and, as $\Gamma' \cap \Omega = \emptyset$, it turns out that $\dim(\langle S_{(r-1)n}, \Omega \rangle \cap \Gamma') = (r - 1)n$, i.e. $\langle S_{(r-1)n}, \Omega \rangle \cap \Gamma'$ is a hyperplane of Γ' . This implies that there exists a point $\bar{P} \in \bar{B} \cap \langle S_{(r-1)n}, \Omega \rangle \cap \Gamma'$ and, as a consequence, $S_{(r-1)n} \cap \langle \bar{P}, \Omega \rangle \neq \emptyset$; i.e. the cone K intersects any $(r - 1)n$ -subspace of Σ' defining an $(r - 1)$ -subspace in Π_r . On the other hand, since $Y \in B$, the hyperplane $PG(\mathcal{S})$ of Π_r intersects B ; so B meets every hyperplane of Π_r .

Suppose now that B contains a line of Π_r .

If $Q = T$, then $B \cap PG(\mathcal{S}) = \{Y\}$ and a line of Π_r contained in B must be defined by an n -dimensional subspace S_n of Σ' with $S_n \cap \Sigma = \{Y\}$. From $S_n \subset B$, we get that the line $\ell = S_n \cap \Gamma'$ through Q is contained in \bar{B} , a contradiction.

If $Q \neq T$, then $|B \cap PG(\mathcal{S})| = q^{n-1} + 1$ and a line contained in B must be defined by an n -dimensional subspace S_n of Σ' such that $S_n \cap \Sigma = X \in \mathcal{S}$, with $X \cap K \neq \emptyset$. If $X = Y$ and ℓ is the line $S_n \cap \Gamma'$, then we have $\ell \setminus \{T\} \subset \bar{B}$, contradicting Property (α). If $X \neq Y$, then $S'_n = \langle S_n, \Omega \rangle \cap \Gamma'$ is an n -dimensional subspace of Γ' containing Q such that $S'_n \setminus \Sigma \subset \bar{B}$. This means that each line of S'_n through the point Q and not in Σ is contained in \bar{B} , a contradiction. \square

We point out that the notations introduced in this section will be used for the rest of the paper, even without explicitly recalling them.

2.1 Construction A

Suppose that \bar{B} is a minimal blocking set of Γ' such that $T = Q$, i.e. suppose that Γ is a tangent hyperplane of \bar{B} at the point Q . In this case, $B = (K \setminus \Sigma) \cup \{Y\}$ and $|B| = |K \setminus \Sigma| + 1 = q^{n-1}(|\bar{B}| - 1) + 1$.

Let $S_{(r-1)n}$ be a hyperplane of Π_r different from $PG(S)$. Note that since $\Gamma' \cap \Omega = \emptyset$, if $S_{(r-1)n}$ contains Y , then $\dim(S_{(r-1)n} \cap \Gamma') = (r - 2)n + 1$ and, if $S_{(r-1)n}$ does not contain Y , then $\dim(\langle S_{(r-1)n}, \Omega \rangle \cap \Gamma') = (r - 1)n$.

Lemma 1 *Let $S_{(r-1)n}$ be a hyperplane of Π_r different from $PG(S)$. If $S_{(r-1)n}$ contains Y and $\bar{S}_{(r-2)n+1}$ is the subspace $S_{(r-1)n} \cap \Gamma'$, then*

$$|B \cap S_{(r-1)n}| = q^{n-1} |(\bar{B} \cap \bar{S}_{(r-2)n+1}) \setminus \{Q\}| + 1. \tag{1}$$

If $S_{(r-1)n}$ does not contain Y and $\bar{S}_{(r-1)n} = \langle S_{(r-1)n}, \Omega \rangle \cap \Gamma'$, then

$$|B \cap S_{(r-1)n}| = |\bar{B} \cap \bar{S}_{(r-1)n}|. \tag{2}$$

Also, equality (2) holds for any hyperplane $\bar{S}_{(r-1)n}$ of Γ' not through Q and for each hyperplane $S_{(r-1)n}$ of Π_r contained in $\langle \Omega, \bar{S}_{(r-1)n} \rangle$.

Proof Equality (1) is straightforward; so assume that $S_{(r-1)n}$ does not contain Y . Then

$$B \cap S_{(r-1)n} = (K \setminus \Sigma) \cap S_{(r-1)n} = \bigcup_{\bar{P} \in \bar{B}} (\langle \bar{P}, \Omega \rangle \cap S_{(r-1)n}).$$

Since $S_{(r-1)n} \cap \Omega = \emptyset$, we have that $\dim(\langle \bar{P}, \Omega \rangle \cap S_{(r-1)n}) \leq 0$ and equality occurs if and only if $\bar{P} \in \bar{B} \cap \bar{S}_{(r-1)n}$. So we can write

$$B \cap S_{(r-1)n} = \bigcup_{\bar{P} \in \bar{B} \cap \bar{S}_{(r-1)n}} (\langle \bar{P}, \Omega \rangle \cap S_{(r-1)n})$$

and since $\langle \bar{P}, \Omega \rangle \cap \langle \bar{P}', \Omega \rangle = \Omega$ for every pair of distinct points $\bar{P}, \bar{P}' \in \bar{B}$, equality (2) follows.

Now, assume that $\bar{S}_{(r-1)n}$ is a hyperplane of Γ' not containing Q and let $H = \langle \bar{S}_{(r-1)n}, \Omega \rangle \cap \Sigma$. Since $\dim H = rn - 2$, by Remark 1, there exists in H a unique \mathcal{S} -subspace $S_{(r-1)n-1}$ of dimension $(r - 1)n - 1$. Also, since $Q \notin \bar{S}_{(r-1)n}$, the spread element Y is not contained in $S_{(r-1)n-1}$ and the $(r - 1)$ -dimensional subspaces of Π_r contained in $\langle \Omega, \bar{S}_{(r-1)n} \rangle$ and skew to Ω are exactly those passing through $S_{(r-1)n-1}$. If $S_{(r-1)n}$ is one of these $(r - 1)$ -dimensional subspaces of Π_r , then $\langle S_{(r-1)n}, \Omega \rangle \cap \Gamma' = \bar{S}_{(r-1)n}$ and equality (2) occurs. \square

Proposition 2 *B is a minimal blocking set of Π_r if and only if \bar{B} is a minimal blocking set of Γ' .*

Proof If B is a minimal blocking set of Π_r , then, by Lemma 1, \bar{B} is a minimal blocking set of Γ' . Now, suppose that \bar{B} is a minimal blocking set of Γ' . As $B \cap PG(S) = \{Y\}$, Y is an essential point of B . Let P be an affine point of B , i.e. $P \in K \setminus \Sigma$ and let \bar{P} be the unique point of \bar{B} such that $P \in \langle \bar{P}, \Omega \rangle$. Since \bar{B} is a minimal blocking set of Γ' , there exists an $((r - 1)n)$ -dimensional subspace $\bar{S}_{(r-1)n}$ of Γ' such that $\bar{S}_{(r-1)n} \cap \bar{B} = \{\bar{P}\}$. Now, consider the hyperplane $\langle \bar{S}_{(r-1)n}, \Omega \rangle$ of Σ' and its intersection with Σ , say $H = \langle \bar{S}_{(r-1)n}, \Omega \rangle \cap \Sigma$. As H is a hyperplane of Σ containing Ω , by Remark 1, H contains an \mathcal{S} -subspace $S_{(r-1)n-1}$ of dimension $(r - 1)n - 1$ not containing Y . By Lemma 1, the join of $S_{(r-1)n-1}$ and P is an $(r - 1)n$ -dimensional subspace $S_{(r-1)n}$ of Σ' meeting B exactly in the point P , i.e. $S_{(r-1)n}$ defines an $(r - 1)$ -dimensional subspace of Π_r tangent to B at P , hence P is an essential point of B . \square

From a t -dimensional blocking set, we obtain via Construction A blocking sets generating projective spaces of dimension at most t , as we prove in the following proposition.

Proposition 3 *If \bar{B} is a t -dimensional blocking set of Γ' , then $\dim\langle B \rangle \leq \min\{r, t\}$.*

Proof Let $\Gamma'_t = \langle \bar{B} \rangle$ ($t \leq (r - 1)n + 1$) and let $\Gamma_{t-1} = \Gamma'_t \cap \Sigma$. Also, let P_1, P_2, \dots, P_{t-1} be $t - 1$ points of Γ_{t-1} , different from Q , such that $\Gamma_{t-1} = \langle Q, P_1, P_2, \dots, P_{t-1} \rangle$, and let X_i be the element of \mathcal{S} containing P_i , for $i = 1, \dots, t - 1$. Then $S^* = \langle Y, X_1, X_2, \dots, X_{t-1} \rangle$ is an \mathcal{S} -subspace of Σ of dimension $nl - 1$, where $l \leq \min\{t, r\}$. Let $S_{nl} = \langle \Gamma'_t, S^* \rangle$ and note that $S_{nl} \cap \Sigma = S^*$, i.e. S_{nl} defines an l -dimensional subspace Π_l of Π_r . Since $K \subset S_{nl}$, the blocking set B is contained in Π_l , and hence $\dim\langle B \rangle \leq l \leq \min\{r, t\}$. \square

Theorem 1 *From a minimal t -dimensional blocking set \bar{B} in a projective space of order q , it is possible to obtain via Construction A minimal r -dimensional blocking sets B in $\Pi_r \cong PG(r, q^n)$ for any r such that $\max\{2, \frac{t-1}{n} + 1\} \leq r \leq t$.*

Proof Let \bar{B} be a minimal t -dimensional blocking set, and let r and n be two integers ($r, n \geq 2$) such that $\max\{2, \frac{t-1}{n} + 1\} \leq r \leq t$. Let $\Pi_r = \Pi_r(\Sigma', \Sigma, \mathcal{S})$ be the Barlotti–Cofman representation of $PG(r, q^n)$. Let Y be a spread element of \mathcal{S} , and let Ω and Q be a hyperplane and a point of Y , respectively, with $Q \notin \Omega$. Let X_1, X_2, \dots, X_{r-1} be $r - 1$ distinct elements of \mathcal{S} such that $\Sigma = \langle Y, X_1, X_2, \dots, X_{r-1} \rangle$. Let $P_j \in X_j$ for $j = 1, \dots, r - 1$ and let Γ' be an $((r - 1)n + 1)$ -dimensional subspace of Σ' disjoint from Ω containing $Q, P_1, P_2, \dots, P_{r-1}$. Since $\frac{t-1}{n} + 1 \leq r \leq t$, there exists a t -dimensional subspace Γ'_t of Γ' through Q such that $P_j \in \Gamma'_t \cap \Sigma = \Gamma_{t-1}$, for any $j = 1, \dots, r - 1$. This means that the smallest \mathcal{S} -subspace containing Γ_{t-1} is Σ . Embed \bar{B} in Γ'_t in such a way that $\bar{B} \cap \Sigma = \{Q\}$, let $K = \bigcup_{\bar{P} \in \bar{B}} \langle \bar{P}, \Omega \rangle$ and let B be the blocking set obtained via Construction A from the cone K . Since $\langle K \rangle = \langle Y, \Gamma'_t \rangle$ and since the smallest \mathcal{S} -subspace containing $\langle K \rangle \cap \Sigma = \langle Y, \Gamma_{t-1} \rangle$ is Σ , we have $\langle B \rangle = \Pi_r$. \square

Now, we describe some three-dimensional minimal blocking sets that can be obtained via Construction A.

Example 1 Some families of minimal three-dimensional blocking sets of $PG(3, q)$ of size $2q + 1, 3q + 1$ ($q > 2$), $4q + 1$ ($q > 2, q$ even), $3q - 1$ ($q > 2$) and $kq + 1$ ($q > 2, q$ even, and $2 \leq k \leq q - 1$) are constructed in [20] and are presented in [15, Sect. 3] as blocking sets of type $\bar{B}_1, \bar{B}_2, \bar{B}_3, \bar{B}_4$ and \bar{B}_5 , respectively. Choosing such blocking sets as base of the cone K , by Theorem 1, we can construct minimal three-dimensional blocking sets B_1, B_2, B_3, B_4 and B_5 of $PG(3, q^n)$, for any $n \geq 2$, of size $2q^n + 1, 3q^n + 1$ ($q > 2$), $4q^n + 1$ ($q > 2, q$ even), $3q^n - 2q^{n-1} + 1$ ($q > 2$) and $kq^n + 1$ ($q > 2, q$ even, and $2 \leq k \leq q - 1$), respectively. It is not difficult to verify that none of such blocking sets is a blocking set of type $\bar{B}_1, \bar{B}_2, \bar{B}_3, \bar{B}_4$, or \bar{B}_5 in $PG(3, q^n)$.

Example 2 In [12], Heim constructed minimal blocking sets in $PG(d, q)$ ($q = p^h$), not contained in a hyperplane, of size $(d - 1)q - (d - 3)q/p + 1$, if $h > 2$, and of size $(d - 1)q - (d - 3)(q + 1)/2 + d - 1$, if $q = p$ is an odd prime. Applying Construction A to such blocking sets, we construct r -dimensional minimal blocking sets in $PG(r, q^n)$ ($q = p^h$), for any $n \geq 2$ and r such that $\max\{2, \frac{d-1}{n} + 1\} \leq r \leq d$, of size $(d - 1)q^n - (d - 3)q^n/p + 1$, if $h > 2$, and of size $(d - 1)q^n - (d - 3)(q^n + q^{n-1})/2 + (d - 2)q^{n-1} + 1$, if $q = p$ is an odd prime.

2.2 Ovoidal blocking sets of type A

We say that a blocking set B obtained via Construction A is an *ovoidal blocking set of type A* if the base \bar{O} of the corresponding cone is an ovoid either of $PG(3, q)$ or of $Q(4, q)$ or of $Q(6, q)$, and we write $B = B_{\bar{O}}$.

Ovoidal blocking sets of type A have a special structure as we show in the following proposition.

Proposition 4 *If $B_{\bar{O}}$ is an ovoidal blocking set of type A in $\Pi_r \cong PG(r, q^n)$ with special point Y , then each secant line through Y meets $B_{\bar{O}}$ in $q^{n-1} + 1$ points. In particular, if $n = 2$, then $B_{\bar{O}}$ is a union of Baer sublines through Y .*

Proof Let S_n be a line of Π_r through Y not contained in $PG(S)$. Then we get

$$S_n \cap B_{\bar{O}} = \cup_{\bar{P} \in S_n \cap \bar{O}} ((\Omega, \bar{P}) \setminus \Omega) \cup \{Y\}$$

and hence $|S_n \cap B_{\bar{O}}| = q^{n-1} |(S_n \cap \bar{O}) \setminus \{Q\}| + 1$. Since $Y \cap \Gamma' = \{Q\}$, $S_n \cap \Gamma' = \ell$ is a line through Q . Hence, since \bar{O} is an ovoid, we get $|S_n \cap B_{\bar{O}}| = 1$ or $q^{n-1} + 1$. If $n = 2$ and $|S_2 \cap B_{\bar{O}}| = q + 1$, then $\ell \cap \bar{O} = \{\bar{P}, Q\}$ and $S_2 \cap B_{\bar{O}} = ((\Omega, \bar{P}) \setminus \Omega) \cup \{Y\}$, and such a set is a Baer subline in the projective space Π_r . □

Let \bar{O} be an ovoid of $PG(3, q)$. By Theorem 1, choosing \bar{O} as base of the cone K , we can construct three-dimensional minimal blocking sets $B_{\bar{O}}$ in $PG(3, q^n)$ of size $q^{n+1} + 1$ for any $n \geq 2$. We call such ovoidal blocking sets of *elliptic type* if \bar{O} is an elliptic quadric and of *Suzuki–Tits type* if \bar{O} is a Suzuki–Tits ovoid. We point out that an ovoidal blocking set of elliptic type and an ovoidal blocking set of Suzuki–Tits type cannot be isomorphic. By Lemma 1, it is easy to verify that a secant plane meets $B_{\bar{O}}$ either in $q + 1$ or in $q^{n-1} + 1$ or in $q^n + 1$ points.

Let \bar{O} be a non-classical ovoid of $Q(4, q)$, $q = p^h$. By Ball [3], \bar{O} is a minimal blocking set of $PG(4, q)$ of size $q^2 + 1$ and hence, by Theorem 1, choosing \bar{O} as base of the cone K , we can obtain minimal blocking sets of size $q^{n+1} + 1$ in $PG(3, q^n)$ and in $PG(4, q^n)$ generating the whole space, for any $n \geq 2$. Since there are four known families of non-classical ovoids of $Q(4, q)$ [13, 22, 23], we obtain with our construction four non-isomorphic families of ovoidal blocking sets of size $q^{n+1} + 1$ of type A in $PG(3, q^n)$ and in $PG(4, q^n)$ generating the whole space, for any $n \geq 2$. We also mention that there is a sporadic ovoid in $Q(4, 3^3)$ which can be used for Construction A [16]. It is not difficult to verify that, by Lemma 1 and by Ball [3], any secant hyperplane meets $B_{\bar{O}}$ in 1 mod p points.

Some of the above results can be summarized in the following theorem.

Theorem 2 *In $PG(3, q^n)$ and in $PG(4, q^n)$, $q = p^h$, p prime, $h \geq 1$, for any $n \geq 2$, there exist non-isomorphic families of minimal blocking sets of size $q^{n+1} + 1$, generating the whole space and intersecting each hyperplane in 1 mod p points. In particular, if $n = 2$, such blocking sets are the union of q^2 Baer sublines through a point.*

Consider the Barlotti–Cofman representation $\Pi_r = \Pi_r(\Sigma', \Sigma, S)$ of $PG(r, q^2)$, $r = 3, 4$, in $\Sigma' = PG(2r, q)$ obtained from a normal line-spread S of $\Sigma = PG(2r - 1, q) \subset \Sigma'$. A hermitian variety $H(r, q^2)$ of Π_r having $PG(S)$ as tangent hyperplane at the point $Y, Y \in S$, is represented in Σ' by a quadratic cone \mathcal{K} with vertex a point Ω of Y and base a hyperbolic quadric $Q^+(5, q)$ if $r = 3$ and an elliptic quadric $Q^-(7, q)$ if $r = 4$. Also, $\mathcal{K} \cap \Sigma$ is a cone with vertex the line Y which can be described as a union of elements of S .

Let $Q(4, q)$ be a parabolic quadric contained in \mathcal{K} intersecting Y in a point Q . If \bar{O} is an ovoid of $Q(4, q)$ such that $\bar{O} \cap \Sigma = \{Q\}$ and K is the cone with vertex Ω and base \bar{O} , then $K \subset \mathcal{K}$ and the corresponding ovoidal blocking set $B_{\bar{O}}$ of Π_r is contained in $H(r, q^2)$, $r = 3, 4$. Note that, if \bar{O} is a non-classical ovoid of $Q(4, q)$, by Theorem 1, we can embed \bar{O} in Σ' in such a way that $B_{\bar{O}}$ generates the whole space. Actually, if $r = 3$, $B_{\bar{O}}$ is a locally

hermitian ovoid of $H(3, q^2)$ with respect to the point Y corresponding to the symplectic spread of $PG(3, q)$ associated, via the inverse of the Plücker map, to the ovoid \bar{O} , and any locally hermitian ovoid of $H(3, q^2)$ associated with a symplectic spread of $PG(3, q)$ can be obtained in this way (see [2, 14] for more details). So we have proved the following theorem.

Theorem 3 *Any locally hermitian ovoid \mathcal{O} of $H(3, q^2)$, $q = p^h$, p prime, $h \geq 1$, associated with a symplectic spread of $PG(3, q)$ is an ovoidal blocking set of type A of $PG(3, q^2)$ and every plane meets \mathcal{O} in $1 \pmod p$ points.*

In the four-dimensional case, we have the following result.

Theorem 4 *In $PG(4, q^2)$, $q = p^h$, p an odd prime, $h > 1$, or $q = 2^{2e+1}$, $e \geq 1$, there exist maximal partial ovoids of $H(4, q^2)$ of size $q^3 + 1$ which are the union of Baer sublines through a point, generating the whole space and intersecting each hyperplane in $1 \pmod p$ points.*

Proof In the above construction, let \bar{O} be a non-classical ovoid of $Q(4, q)$. Since $|\bar{O}| = q^2 + 1$, the blocking set $B_{\bar{O}}$ has size $q^3 + 1$ and, by Proposition 4, it is the union of q^2 chords (Baer sublines contained in $H(4, q^2)$) through Y . Let R and S be two distinct points of $B_{\bar{O}}$ and suppose that the line π_{RS} of Π_4 through R and S is a generator of $H(4, q^2)$. Since $B_{\bar{O}}$ is a union of chords of $H(4, q^2)$ through Y , the line π_{RS} does not contain Y and hence π_{RS} is a plane of Σ' contained in the quadratic cone \mathcal{K} intersecting Σ in a line of S . Let R', S' be the points of \bar{O} such that $R' \in \langle \Omega, R \rangle$ and $S' \in \langle \Omega, S \rangle$. Since the line of Σ' joining R and S is contained in \mathcal{K} , the plane $\pi = \langle \Omega, R, S \rangle$ is contained in \mathcal{K} and hence the line joining R' and S' in Σ' is contained in $Q(4, q)$, a contradiction. Hence, two points of $B_{\bar{O}}$ are not contained in a generator of $H(4, q^2)$, i.e. $B_{\bar{O}}$ is a partial ovoid of $H(4, q^2)$.

Now suppose, by way of contradiction, that the partial ovoid $B_{\bar{O}}$ is not maximal. This means that there exists a point $P \in H(4, q^2) \setminus B_{\bar{O}}$ such that $B_{\bar{O}} \cup \{P\}$ is a partial ovoid of $H(4, q^2)$. Then the tangent hyperplane to $H(4, q^2)$ at the point P does not contain any point of the blocking set $B_{\bar{O}}$, a contradiction. \square

Finally, let \bar{O} be an ovoid of $Q(6, q)$. By Ball and Govaerts [4], \bar{O} is a six-dimensional minimal blocking set of $PG(6, q)$ of size $q^3 + 1$, hence, by choosing \bar{O} as base of the cone K , we obtain a family of minimal blocking sets of size $q^{n+2} + 1$ of $PG(r, q^n)$, with $3 \leq r \leq 6$ if $n \geq 3$, and of $PG(r, q^2)$, with $4 \leq r \leq 6$, generating the whole space. Since the known families of ovoids of $Q(6, q)$ are the Thas–Kantor ovoids with $q = 3^h$ and $h \geq 1$, and the Ree–Tits ovoids with $q = 3^{2h+1}$, $h > 0$ (see for instance [13, 21]), we obtain with our construction two non-isomorphic families of ovoidal blocking sets of type A in characteristic 3. Also in this case, by Lemma 1 and by Ball and Govaerts [4], a secant hyperplane meets $B_{\bar{O}}$ in $1 \pmod p$ points. Hence, we have

Theorem 5 *In $PG(r, q^n)$, for $3 \leq r \leq 6$, $n \geq 3$ and q a power of 3, there exist minimal blocking sets of size $q^{n+2} + 1$ generating the whole space and intersecting each hyperplane in $1 \pmod three$ points. Also, there exist in $PG(r, q^2)$, $4 \leq r \leq 6$, q a power of 3, minimal blocking sets of size $q^4 + 1$ generating the whole space, and which are the union of q^3 Baer sublines through a point.*

In some cases the ovoidal blocking sets of $PG(r, q^2)$ obtained from ovoids of $Q(6, q)$ can be embedded in a hermitian variety of $PG(r, q^2)$.

Consider the Barlotti–Cofman representation $\Pi_r = \Pi_r(\Sigma', \Sigma, S)$ of $PG(r, q^2)$, $4 \leq r \leq 6$, in $\Sigma' = PG(2r, q)$ obtained from a normal line-spread \mathcal{S} of $\Sigma = PG(2r - 1, q) \subset \Sigma'$. A hermitian variety $H(r, q^2)$ of Π_r having $PG(S)$ as tangent hyperplane at the point Y , $Y \in \mathcal{S}$, is represented in Σ' by a quadratic cone \mathcal{K} with vertex a point Ω of Y and base an elliptic quadric $Q^-(7, q)$ of Σ' if $r = 4$, a hyperbolic quadric $Q^+(9, q)$ if $r = 5$, and an elliptic quadric $Q^-(11, q)$ if $r = 6$. Also, $\mathcal{K} \cap \Sigma$ is a cone with vertex the line Y which can be described as a union of elements of \mathcal{S} .

Let $Q(6, q)$ be a parabolic quadric contained in \mathcal{K} , and intersecting Y in a point Q . If \bar{O} is an ovoid of $Q(6, q)$ such that $\bar{O} \cap \Sigma = \{Q\}$ and K is the cone with vertex Ω and base \bar{O} , then $K \subset \mathcal{K}$ and the ovoidal blocking set $B_{\bar{O}}$ of Π_r obtained from the cone K is contained in $H(r, q^2)$. Also in this case, by Theorem 1, we can embed \bar{O} in Σ' in such a way that $B_{\bar{O}}$ generates the whole space and we have the following result.

Theorem 6 *The minimal blocking set $B_{\bar{O}}$ is a maximal partial ovoid of $H(r, q^2)$, $4 \leq r \leq 6$, of size $q^4 + 1$ which is the union of q^3 Baer sublines through a point Y , generating the whole space and intersecting each hyperplane in $1 \pmod p$ points.*

Proof The result follows as in the proof of Theorem 4. □

Corollary 1 *There exist maximal partial ovoids of $H(r, q^2)$, $4 \leq r \leq 6$ and q a power of 3, of size $q^4 + 1$ which are the union of q^3 Baer sublines through a point.*

2.3 Construction B

Suppose that $Q \notin Y$, let Z be the unique element of \mathcal{S} such that $Q \in Z$ and let $\Gamma \cap Y = \{T\}$. In this case, the size of B is given by

$$|B| = q^{n-1}(|\bar{B}| - 1) + q^{n-1} + 1 = q^{n-1}|\bar{B}| + 1$$

also $B \cap PG(S)$ is contained in the line $\langle Y, Z \rangle$ of $PG(S)$ and $|B \cap PG(S)| = q^{n-1} + 1$. The intersection numbers of B with respect to the hyperplanes can be determined as in Construction A.

Recall that if $S_{(r-1)n}$ is a hyperplane of Π_r other than $PG(S)$ containing Y , then $\dim(S_{(r-1)n} \cap \Gamma') = (r - 2)n + 1$. Whereas if $S_{(r-1)n}$ is a hyperplane of Π_r not containing Y , then $\dim(\langle S_{(r-1)n}, \Omega \rangle \cap \Gamma') = (r - 1)n$.

Lemma 2 *Let $S_{(r-1)n}$ be a hyperplane of Π_r other than $PG(S)$. If $S_{(r-1)n}$ contains Y and $\bar{S}_{(r-2)n+1}$ denotes the $((r - 2)n + 1)$ -dimensional subspace through T obtained intersecting $S_{(r-1)n}$ and Γ' , then*

$$|B \cap S_{(r-1)n}| = q^{n-1}|\bar{S}_{(r-2)n+1} \cap \bar{B}| + 1. \tag{3}$$

If $S_{(r-1)n}$ does not contain Y and $\bar{S}_{(r-1)n} = \langle S_{(r-1)n}, \Omega \rangle \cap \Gamma'$, then

$$|B \cap S_{(r-1)n}| = |\bar{B} \cap \bar{S}_{(r-1)n}|. \tag{4}$$

Also, equality (4) holds for any hyperplane $\bar{S}_{(r-1)n}$ of Γ' not through T and for each of the q^{n-1} hyperplanes $S_{(r-1)n}$ of Π_r contained in $\langle \Omega, \bar{S}_{(r-1)n} \rangle$.

Proof Let $S_{(r-1)n}$ be a hyperplane of Π_r , other than $PG(S)$, containing Y and let $\bar{S}_{(r-2)n+1} = S_{(r-1)n} \cap \Gamma'$. Then we have

$$B \cap S_{(r-1)n} = ((K \setminus \Sigma) \cap S_{(r-1)n}) \cup \{X \in \mathcal{S} : X \cap K \cap S_{(r-1)n} \neq \emptyset\}.$$

Since

$$(K \setminus \Sigma) \cap S_{(r-1)n} = \bigcup_{\bar{P} \in (\bar{B} \cap \bar{S}_{(r-2)n+1}) \setminus \{Q\}} (\langle \bar{P}, \Omega \rangle \setminus \Omega)$$

and $|\{X \in S : X \cap K \cap S_{(r-1)n} \neq \emptyset\}| = 1$ if $Q \notin \bar{S}_{(r-2)n+1}$ and $q^{n-1} + 1$ otherwise, we get equality (3). If $S_{(r-1)n}$ is a hyperplane of Π_r not containing Y , with the same arguments of Lemma 1, we obtain the result. \square

Remark 2 Note that if $S_{(r-1)n}$ is a hyperplane of Π_r passing through a point of $B \cap PG(S)$ different from Y , then $Q \in \bar{S}_{(r-1)n} = \langle S_{(r-1)n}, \Omega \rangle \cap \Gamma'$.

Proposition 5 *Let P be an affine point of B and let $\langle \bar{P} \rangle = \langle P, \Omega \rangle \cap \Gamma'$. The point P is an essential point of B if and only if there exists in Γ' a tangent hyperplane to \bar{B} at the point \bar{P} , not passing through T .*

Proof If $S_{(r-1)n}$ is a hyperplane of Π_r tangent to B at the point P , then by Lemma 2, $\bar{S}_{(r-1)n} = \langle S_{(r-1)n}, \Omega \rangle \cap \Gamma'$ is a hyperplane, not through T , of Γ' tangent to \bar{B} at the point \bar{P} . Conversely, if there exists a tangent hyperplane $\bar{S}_{(r-1)n}$, not through T , to \bar{B} at the point \bar{P} , by Lemma 2, there exist at least q^{n-1} hyperplanes of Π_r tangent to B at the point P . \square

The above proposition shows that the minimality of B does not automatically follow from the minimality of \bar{B} , as in Construction A; to this end, we need some extra conditions on \bar{B} . We say that \bar{B} satisfies Condition (*) with respect to the point T if:

(*) for each point $\bar{P} \in \bar{B} \setminus \{Q\}$, there exists a tangent hyperplane to \bar{B} passing through \bar{P} , but not containing T .

As a corollary of Proposition 5, we get

Corollary 2 *The affine points of B are essential points if and only if \bar{B} satisfies Condition (*) with respect to the point T .*

In the next proposition, we prove that if \bar{B} does not span the whole space Γ' , then it is possible to embed \bar{B} in Γ' in such a way that all affine points of B are essential points.

Proposition 6 *If \bar{B} is a minimal blocking set of Γ' contained in a t -dimensional subspace Γ'_t of Γ' , with $t \leq (r - 1)n$ and $T \notin \Gamma'_t$, then all affine points of B are essential points.*

Proof Since $T \notin \Gamma'_t$, for each point $\bar{P} \in \bar{B} \setminus \{Q\}$, there exists a hyperplane of Γ' tangent to \bar{B} at the point \bar{P} which does not contain T , i.e. \bar{B} satisfies Condition (*) with respect to the point T . \square

Now, let Γ_n be the n -dimensional subspace of Γ obtained intersecting $\langle Y, Z \rangle$ and Γ . Let X be a point of $B \cap PG(S)$ different from Y and let $\Gamma_{n-1}^X = \langle X, \Omega \rangle \cap \Gamma_n$. It is clear that Γ_{n-1}^X is a hyperplane of Γ_n through Q not containing the point T . By Lemma 2, the intersection numbers of B with respect to the hyperplanes of Π_r through X , not containing Y , are determined by the intersection numbers of \bar{B} with respect to the hyperplanes of Γ' , different from Γ , containing Γ_{n-1}^X . Indeed, if $S_{(r-1)n}$ contains X and does not contain Y , then $\bar{S}_{(r-1)n}$ contains Γ_{n-1}^X . Conversely, if Γ_{n-1} is a hyperplane of Γ_n passing through Q and not containing T , then the intersection numbers of \bar{B} with respect to the hyperplanes of Γ' , different from Γ , containing Γ_{n-1} , determine the intersection numbers of B with respect to the hyperplanes of Π_r (not containing Y) through the unique element X of $S \setminus \{Y\}$ contained in $\langle \Gamma_{n-1}, \Omega \rangle$. Hence, we have:

Proposition 7 *Let $X \in B \cap PG(S)$, with $X \neq Y$. The point X is an essential point of B if and only if there exists a hyperplane of Γ' , different from Γ , tangent to \bar{B} at the point Q and containing the subspace $\Gamma_{n-1}^X = \langle X, \Omega \rangle \cap \Gamma_n$. Also, the number of essential points of B on $PG(S) \setminus \{Y\}$ equals the number of hyperplanes of Γ_n passing through the point Q , not containing the point T and contained in a tangent hyperplane to \bar{B} different from Γ .*

Denote by Γ'_t the t -dimensional subspace spanned by \bar{B} and let $\Gamma_{t-1} = \Gamma'_t \cap \Gamma$. Since \bar{B} is contained in Γ'_t , it is a blocking set of Γ'_t with respect to the hyperplanes. From now on, we suppose that $T \notin \Gamma_{t-1}$ (hence $t \leq (r - 1)n$), so by Proposition 6, all the affine points of B are essential points.

Let $\Pi_{r-2} = S_{(r-1)n-1}$ be a hyperplane of $PG(S)$ through Y , not containing Z , let $\Gamma^{\Pi_{r-2}} = S_{(r-1)n-1} \cap \Gamma_{t-1}$, and note that $\dim \Gamma^{\Pi_{r-2}} \geq t - n - 1$.

Proposition 8

- (●) *If there exists a hyperplane $\Pi_{r-2} = S_{(r-1)n-1}$ of $PG(S)$ through Y , but not containing Z , such that $\dim \Gamma^{\Pi_{r-2}} \geq t - n$, then Y is an essential point of B . In particular, if $t \leq n$ and $\Gamma_{t-1} \not\subseteq \Gamma_n$, then Y is an essential point of B .*
- (●●) *If there exists a hyperplane $\Pi_{r-2} = S_{(r-1)n-1}$ of $PG(S)$ through Y , not containing Z , such that $\dim \Gamma^{\Pi_{r-2}} = t - n - 1$ and such that $\Gamma^{\Pi_{r-2}}$ is contained in a $(t - n)$ -dimensional subspace Γ'_{t-n} of Γ'_t , not contained in Σ , and disjoint from \bar{B} , then Y is an essential point of B .*

Proof

(●) Let $\Pi_{r-2} = S_{(r-1)n-1}$ be the hyperplane of $PG(S)$ such that $\dim \Gamma^{\Pi_{r-2}} = s \geq t - n$ and let $\Pi_{r-1} = S_{(r-1)n}$ be a hyperplane of Π_r , different from $PG(S)$, containing Π_{r-2} . By Lemma 2, since $\bar{B} \subset \Gamma'_t$ and since $S_{(r-1)n} \cap \Gamma'_t \cap \Sigma = \Gamma^{\Pi_{r-2}}$, we have $|\Pi_{r-1} \cap B| = q^{n-1} |S^{\Pi_{r-1}} \cap \bar{B}| + 1$, where $S^{\Pi_{r-1}} = S_{(r-1)n} \cap \Gamma'_t$ and $s \leq \dim S^{\Pi_{r-1}} \leq s + 1$. If for all Π_{r-1} different from $PG(S)$ containing Π_{r-2} , the dimension of $S^{\Pi_{r-1}}$ is $s + 1$, then $|\Gamma'_t \setminus \Sigma| = q^t = q^n \cdot q^{s+1}$, and hence $s = t - n - 1$, a contradiction. Then there exists a hyperplane $\bar{\Pi}_{r-1} = \bar{S}_{(r-1)n}$ of Π_r containing Π_{r-2} such that $\dim(\bar{S}_{(r-1)n} \cap \Gamma'_t) = s$, i.e. $\bar{S}_{(r-1)n} \cap \Gamma'_t = \Gamma^{\Pi_{r-2}}$ and hence $S^{\bar{\Pi}_{r-1}} \cap \bar{B} = \emptyset$. This means that $\bar{\Pi}_{r-1}$ is a tangent hyperplane to B at the point Y .

If $t \leq n$ and $\Gamma_{t-1} \not\subseteq \Gamma_n$, it is possible to find a hyperplane Π_{r-2} of $PG(S)$ through Y , but not containing Z , such that $\dim \Gamma^{\Pi_{r-2}} \geq 0 \geq t - n$. So the preceding arguments show that Y is an essential point of B .

(●●) In this case, the hyperplane Π_{r-1} of Π_r obtained joining Π_{r-2} and Γ'_{t-n} is a tangent hyperplane to B at the point Y . □

If B' is a minimal blocking set of Π_r contained in B , by Corollary 1, Propositions 7 and 8, we get

Corollary 3 *If \bar{B} satisfies Condition (*) with respect to the point T , then*

$$|B'| = q^{n-1} (|\bar{B}| - 1) + t_Q + \varepsilon,$$

where t_Q is the number of hyperplanes of Γ_n passing through Q , not containing T and contained in a tangent hyperplane to \bar{B} different from Γ ; whereas $\varepsilon = 1$ if Y is an essential point of B and $\varepsilon = 0$ otherwise. Also, if $t \leq n$, then $\varepsilon = 1$.

Proposition 9

- (1) If $t \leq (r - 2)n + 1$ and $\Gamma_n \cap \Gamma_{t-1} = \{Q\}$, then each point $X \in B \cap PG(S)$, different from Y , is an essential point of B . Hence, $|B'| = q^{n-1}|\bar{B}| + \varepsilon$, where $\varepsilon = 0$ or 1 .
- (2) If $(r - 2)n + 1 < t \leq (r - 1)n$ and $\dim(\Gamma_n \cap \Gamma_{t-1}) = t - (r - 2)n - 1$, then the number of essential points of B on $PG(S)$, different from Y , is at least $q^{(r-1)n-t}$. Hence, $|B'| \geq q^{n-1}(|\bar{B}| - 1) + q^{(r-1)n-t} + \varepsilon$, where $\varepsilon = 0$ or 1 .

Proof

- (1) Let $t \leq (r - 2)n + 1$. In this case, we may assume that $\Gamma_n \cap \Gamma_{t-1} = \{Q\}$. Let $X \in B \cap PG(S)$ be different from Y and let $\Gamma_{n-1}^X = \langle X, \Omega \rangle \cap \Gamma_n$ (recall that $\Gamma_n = \Gamma \cap \langle Y, Z \rangle$). Then

$$\dim \langle \Gamma_{n-1}^X, \Gamma_{t-1} \rangle \leq (r - 1)n - 1$$

and $T \notin \langle \Gamma_{n-1}^X, \Gamma_{t-1} \rangle$. Hence, there exists a hyperplane $\Gamma_{(r-1)n-1}$ of Γ not containing T and containing $\langle \Gamma_{n-1}^X, \Gamma_{t-1} \rangle$. Now, if $P \in \Gamma' \setminus (\Sigma \cup \Gamma'_t)$, then the space joining $\Gamma_{(r-1)n-1}$ and P is a hyperplane of Γ' , different from Γ , tangent to \bar{B} at the point Q , and containing the subspace Γ_{n-1}^X , thus, by Proposition 7, X is an essential point of B . Hence, since $T \notin \Gamma'_t$, all points of $B \setminus \{Y\}$ are essential points, i.e. $|B'| = q^{n-1}|\bar{B}| + \varepsilon$, where $\varepsilon = 0$ or 1 .

- (2) Now, let $t > (r - 2)n + 1$, let $\dim(\Gamma_n \cap \Gamma_{t-1}) = t - (r - 2)n - 1$, put $s = t - (r - 2)n - 1$ and $\Gamma_s = \Gamma_n \cap \Gamma_{t-1}$. If Γ_{n-1} is a hyperplane of Γ_n not containing T and containing Γ_s , then $\langle \Gamma_{n-1}, \Gamma_{t-1} \rangle$ is a hyperplane of Γ not containing T and containing Γ_{n-1} . Now, as in the previous case, since $t \leq (r - 1)n$, there exists a hyperplane of Γ' , different from Γ , tangent to \bar{B} at Q and containing Γ_{n-1} . Hence, by Proposition 7, the number of essential points of B on $PG(S) \setminus \{Y\}$ is at least the number of hyperplanes of Γ_n containing Γ_s , but not containing T . Hence, $|B'| \geq q^{n-1}(|\bar{B}| - 1) + q^{(r-1)n-t} + \varepsilon$, where $\varepsilon = 0$ or 1 . □

Proposition 10 *Suppose that there exists a hyperplane $\bar{\Gamma}_{t-1}$, different from Γ_{t-1} , of Γ'_t tangent to \bar{B} at the point Q and let $\bar{\Gamma}_{t-2} = \Sigma \cap \bar{\Gamma}_{t-1}$. Then the following results hold.*

- (1') If $t \leq (r - 2)n + 2$ and $\Gamma_n \cap \bar{\Gamma}_{t-2} = \{Q\}$, then each point $X \in B \cap PG(S)$, different from Y , is an essential point of B . Hence, $|B'| = q^{n-1}|\bar{B}| + \varepsilon$, where $\varepsilon = 0$ or 1 .
- (2') If $(r - 2)n + 2 < t \leq (r - 1)n$ and $\dim(\Gamma_n \cap \bar{\Gamma}_{t-2}) = t - (r - 2)n - 2$, then the number of essential points of B on $PG(S)$, different from Y , is at least $q^{(r-1)n-t+1}$. Hence, $|B'| \geq q^{n-1}(|\bar{B}| - 1) + q^{(r-1)n-t+1} + \varepsilon$, where $\varepsilon = 0$ or 1 .

Proof

- (1') Let $t \leq (r - 2)n + 2$ and suppose that $\Gamma_n \cap \bar{\Gamma}_{t-2} = \{Q\}$. Let $X \in B \cap PG(S)$ be different from Y and let $\Gamma_{n-1}^X = \langle X, \Omega \rangle \cap \Gamma_n$. As in case (1), there exists a hyperplane $\Gamma_{(r-1)n-1}$ of Γ not containing T and containing $\langle \Gamma_{n-1}^X, \bar{\Gamma}_{t-2} \rangle$. Now, the space joining $\Gamma_{(r-1)n-1}$ and $\bar{\Gamma}_{t-1}$ is a hyperplane of Γ' , different from Γ , tangent to \bar{B} at the point Q and containing the subspace Γ_{n-1}^X , thus, by Proposition 7, X is an essential point of B . Hence, $|B'| = q^{n-1}|\bar{B}| + \varepsilon$, where $\varepsilon = 0$ or 1 .

- (2') Now, let $t > (r - 2)n + 2$, let $\dim(\Gamma_n \cap \bar{\Gamma}_{t-2}) = t - (r - 2)n - 2$, put $s' = t - (r - 2)n - 2$ and $\Gamma_{s'} = \Gamma_n \cap \bar{\Gamma}_{t-2}$. If Γ_{n-1} is a hyperplane of Γ_n not containing T and containing $\Gamma_{s'}$, then

$$\dim \langle \Gamma_{n-1}, \bar{\Gamma}_{t-2} \rangle = n - 1 + t - 2 - s' = (r - 1)n - 1$$

and, since $T \notin \Gamma_{n-1}$, $T \notin \langle \Gamma_{n-1}, \bar{\Gamma}_{t-2} \rangle$. Hence, there exists a hyperplane $\Gamma_{(r-1)n-1}$ of Γ not containing T and containing $\langle \Gamma_{n-1}^X, \bar{\Gamma}_{t-2} \rangle$. Now, using the same arguments of case (1)', we obtain the desired result. \square

From a given t -dimensional blocking set, we can obtain via Construction B blocking sets generating subspaces of dimension at most $t + 1$, as we prove in the following proposition.

Proposition 11 *If \bar{B} is a t -dimensional blocking set, then $\dim\langle B \rangle \leq \min\{r, t + 1\}$.*

Proof Let $\Gamma'_t = \langle \bar{B} \rangle$, $t \leq (r - 1)n + 1$, and let $\Gamma_{t-1} = \Gamma'_t \cap \Sigma$. Also, let P_1, P_2, \dots, P_t be t points of Γ_{t-1} such that $\Gamma_{t-1} = \langle P_1, P_2, \dots, P_t \rangle$ and let X_i be the element of \mathcal{S} containing P_i , for $i = 1, \dots, t$. Then $S^* = \langle Y, X_1, X_2, \dots, X_t \rangle$ is an \mathcal{S} -subspace of Σ of dimension $nl - 1$, where $l \leq \min\{t + 1, r\}$. Let $S_{nl} = \langle \Gamma'_t, S^* \rangle$ and note that $S_{nl} \cap \Sigma = S^*$, i.e. S_{nl} defines an l -dimensional subspace, say Π_l , of Π_r . Also, since $K \subset S_{nl}$, the blocking set B is contained in Π_l , i.e. $\dim\langle B \rangle \leq l \leq \min\{r, t + 1\}$. \square

In the next theorem, we determine the minimal blocking sets that can be obtained via Construction B.

Theorem 7 *From a minimal t -dimensional blocking set \bar{B} in a projective space of order q , we can obtain via Construction B minimal r -dimensional blocking sets B' in $PG(r, q^n)$ for any r such that $\max\{2, \frac{t}{n} + 1\} \leq r \leq t + 1$. Also,*

- (a) $|B'| = q^{n-1}|\bar{B}| + 1$ (i.e. $B' = B$) if $t \leq (r - 2)n + 1$;
- (b) $|B'| = q^{n-1}|\bar{B}| + \varepsilon$ ($\varepsilon = 0, 1$) if $t = (r - 2)n + 2$ and there exist at least two tangent hyperplanes to \bar{B} at the point Q ;
- (c) $|B'| \geq q^{n-1}(|\bar{B}| - 1) + q^{(r-1)n-t} + \varepsilon$ ($\varepsilon = 0, 1$) if $(r - 2)n + 1 < t \leq (r - 1)n$;
- (d) $|B'| \geq q^{n-1}(|\bar{B}| - 1) + q^{(r-1)n-t+1} + \varepsilon$ ($\varepsilon = 0, 1$) if $(r - 2)n + 2 < t \leq (r - 1)n$ and there exist at least two tangent hyperplanes to \bar{B} at the point Q .

Proof Let \bar{B} be a minimal t -dimensional blocking set, and let r and n be two integers, $r, n \geq 2$, such that $\max\{2, \frac{t}{n} + 1\} \leq r \leq t + 1$. Let Π_r be the r -dimensional projective space of order q^n obtained via the Barlotti–Cofman construction described in Sect. 2. Let $Y, Z, X_1, \dots, X_{r-2}$ be r elements of \mathcal{S} such that $\Sigma = \langle Y, Z, X_1, \dots, X_{r-2} \rangle$, let T, Q, P_j ($j = 1, \dots, r - 2$) be r points of Σ such that $T \in Y, Q \in Z$, and $P_j \in X_j$ for $j = 1, \dots, r - 2$, and let Ω be a hyperplane of Y not containing T . Also, let Γ' be an $((r - 1)n + 1)$ -dimensional subspace of Σ' disjoint from Ω , containing $T, Q, P_1, \dots, P_{r-2}$.

Since $\frac{t}{n} + 1 \leq r \leq t + 1$, there exists a t -dimensional subspace Γ'_t of Γ' not contained in Σ , such that $Q, P_j \in \Gamma'_t \cap \Sigma = \Gamma_{t-1}$ for $j = 1, \dots, r - 2$, and such that $T \notin \Gamma'_t$. This means that the smallest \mathcal{S} -subspace containing Γ_{t-1} is Σ . Embed \bar{B} in Γ'_t in such a way that $\bar{B} \cap \Sigma = \{Q\}$ and let $K = \bigcup_{\bar{P} \in \bar{B}} \langle \bar{P}, \Omega \rangle$. Since $\langle K \rangle = \langle \Omega, \Gamma'_t \rangle$ and since the smallest \mathcal{S} -subspace containing $\langle K \rangle \cap \Sigma = \langle \Omega, \Gamma_{t-1} \rangle$ is Σ , we have $\langle B \rangle = \Pi_r$. By construction, $T \notin \Gamma'_t$, hence all the affine points of B are essential points, so if B' is the minimal blocking set contained in B , we have that $\langle B \rangle = \langle B' \rangle$.

If $t \leq (r - 2)n + 1$, we can choose Γ_{t-1} in such a way that $\Gamma_{t-1} \cap \Gamma_n = \{Q\}$ and hence, by Proposition 9 case (1), $|B'| = q^{n-1}|\bar{B}| + \varepsilon$ ($\varepsilon = 0, 1$). Also, if $t \leq n$, then $\varepsilon = 1$ (Proposition 8). If $t > n$ and there exists a hyperplane Π_{r-2} of $PG(\mathcal{S})$ through Y , not containing Z , for which $\dim \Gamma^{\Pi_{r-2}} \geq t - n$, then Y is an essential point of B (Proposition 8 (•)). If $t > n$ and for all hyperplanes Π_{r-2} of $PG(\mathcal{S})$ through Y , not containing Z , $\dim \Gamma^{\Pi_{r-2}} = t - n - 1$, we can embed \bar{B} in Γ'_t in such a way that there exists a hyperplane $\Pi_{r-2} = S_{(r-1)n-1}$ of $PG(\mathcal{S})$ through Y , not containing Z , such that $\dim \Gamma^{\Pi_{r-2}} = t - n - 1$ and $\Gamma^{\Pi_{r-2}}$ is contained in a

$(t - n)$ -dimensional subspace Γ'_{t-n} of Γ'_t , not contained in Σ , disjoint from \bar{B} . So, by case (••) of Proposition 8, $|B'| = q^{n-1}|\bar{B}| + 1$. This ends the discussion of case (a).

If $(r - 2)n + 1 < t \leq (r - 1)n$, we can choose Γ_{t-1} in such a way that $\dim(\Gamma_{t-1} \cap \Gamma_n) = t - [(r - 2)n + 1]$ and hence, by case (2) of Proposition 9, $|B'| \geq q^{n-1}(|\bar{B}| - 1) + q^{(r-1)n-t} + \varepsilon$ ($\varepsilon = 0, 1$). This treats case (c).

Now, suppose that there exists in Γ'_t a tangent hyperplane $\bar{\Gamma}_{t-1}$ to \bar{B} at the point Q different from Γ_{t-1} , and let $\bar{\Gamma}_{t-2} = \bar{\Gamma}_{t-1} \cap \Gamma_{t-1}$. In this case, if $t \leq (r - 2)n + 2$, we can choose Γ_{t-1} in such a way that $\bar{\Gamma}_{t-2} \cap \Gamma_n = \{Q\}$, so by case (1)' of Proposition 10, $|B'| = q^{n-1}|\bar{B}| + \varepsilon$ ($\varepsilon = 0, 1$), leading to case (b). While, if $(r - 2)n + 2 < t \leq (r - 1)n$, we can choose Γ_{t-1} in such a way that $\dim(\bar{\Gamma}_{t-2} \cap \Gamma_n) = t - [(r - 2)n + 2]$ and hence by case (2)' of Proposition 10, $|B'| \geq q^{n-1}(|\bar{B}| - 1) + q^{(r-1)n-t+1} + \varepsilon$ ($\varepsilon = 0, 1$). This concludes the proof of case (d). □

Applying Theorem 7, it is possible to construct a lot of examples of non-planar minimal blocking sets, but we will only describe in detail the non-planar minimal blocking sets that can be constructed starting from ovoids.

2.4 Ovoidal blocking sets of type B

We say that a blocking set B obtained via Construction B is an *ovoidal blocking set of type B* if the base \bar{O} of the corresponding cone is an ovoid either of $PG(3, q)$ or of $Q(4, q)$ or of $Q(6, q)$, and we write $B = B_{\bar{O}}$.

Proposition 12 *Let $B_{\bar{O}}$ be an ovoidal blocking set of type B in $\Pi_r \cong PG(r, q^n)$ with special point Y , $t \leq (r - 1)n$ and $T \notin \Gamma'_t$. Then each secant line through Y meets $B_{\bar{O}}$ in $q^{n-1} + 1$ points. In particular, if $n = 2$, then $B_{\bar{O}}$ is a union of Baer sublines through Y .*

Proof Let S_n be a line of Π_r through Y not contained in $PG(S)$. Then we get

$$S_n \cap B_{\bar{O}} = \cup_{\bar{P} \in S_n \cap \bar{O}} ((\Omega, \bar{P}) \setminus \Omega) \cup \{Y\}$$

and hence $|S_n \cap B_{\bar{O}}| = q^{n-1}|S_n \cap \bar{O}| + 1$. Since $Y \cap \Gamma'_t = \emptyset$, then $|S_n \cap \Gamma'_t| \leq 1$ and so $|S_n \cap B_{\bar{O}}| = 1$ or $q^{n-1} + 1$. If $n = 2$ and $|S_2 \cap B_{\bar{O}}| = q + 1$, then $S_2 \cap B_{\bar{O}}$ is a Baer subline in the projective space Π_r (see proof of Proposition 4). Finally, $B_{\bar{O}} \cap PG(S) = B_{\bar{O}} \cap \langle Y, Z \rangle$ and $|B_{\bar{O}} \cap \langle Y, Z \rangle| = q^{n-1} + 1$. Also, if $n = 2$, $B_{\bar{O}} \cap \langle Y, Z \rangle$ is the set of lines of \mathcal{S} intersecting the line $\langle \Omega, Q \rangle$, i.e. $B_{\bar{O}} \cap \langle Y, Z \rangle$ is a Baer subline of $\Pi_r \cong PG(r, q^2)$. □

Let \bar{O} be an ovoid of $PG(3, q)$. By Theorem 7, choosing \bar{O} as base of the cone K , we can construct minimal blocking sets $B_{\bar{O}}$ in $PG(3, q^n)$ and $PG(4, q^n)$, of size $q^{n+1} + q^{n-1} + 1$, generating the whole space. By Lemma 2, it is easy to verify that a secant hyperplane meets $B_{\bar{O}}$ either in $q + 1$ or in $q^{n-1} + 1$ or in $2q^{n-1} + 1$ or in $q^n + q^{n-1} + 1$ points. Hence, the following holds true.

Theorem 8 *There exist in $PG(3, q^n)$ and in $PG(4, q^n)$ for any q and for any $n \geq 2$, minimal blocking sets of size $q^{n+1} + q^{n-1} + 1$ generating the whole space, with intersection numbers with respect to the hyperplanes in the set $\{1, q + 1, q^{n-1} + 1, 2q^{n-1} + 1, q^n + q^{n-1} + 1\}$. In particular, if $n = 2$ such blocking sets are the union of $q^2 + 1$ Baer sublines through a point.*

Now, let \bar{O} be a non-classical ovoid of $Q(4, q)$, $q = p^h$. By Ball [3], \bar{O} is a minimal blocking set of $PG(4, q)$ of size $q^2 + 1$ and hence, by Theorem 7, choosing \bar{O} as base of the cone K we can construct the following minimal blocking sets generating the whole space:

- (1) $B' = B_{\bar{O}}$ of size $q^{n+1} + q^{n-1} + 1$ in $PG(3, q^n)$, $PG(4, q^n)$ and $PG(5, q^n)$ if $n \geq 3$, and in $PG(4, q^n)$ and $PG(5, q^n)$ if $n = 2$.
- (2) B' of size $q^3 + q + \varepsilon$ ($\varepsilon = 0$ or 1) in $PG(3, q^2)$ if there exist at least two tangent hyperplanes to \bar{O} at the point Q , and B' of size at least $q^3 + 1 + \varepsilon$ ($\varepsilon = 0$ or 1) in $PG(3, q^2)$ otherwise.

Applying the above constructions to the known examples of non-classical ovoids of $Q(4, q)$, we obtain the following result.

Theorem 9 *In $PG(r, q^n)$, $3 \leq r \leq 5$ and $n \geq 3$, and in $PG(r, q^2)$, $4 \leq r \leq 5$, with either $q = p^h$, p an odd prime and $h > 1$, or $q = 2^{2e+1}$, $e \geq 1$, there exist r -dimensional minimal blocking sets of size $q^{n+1} + q^{n-1} + 1$ which are the union of $(q^{n-1} + 1)$ -secants through a point. In particular, if $n = 2$, such blocking sets are the union of $q^2 + 1$ Baer sublines through a point.*

Finally, let \bar{O} be an ovoid of $Q(6, q)$, $q = p^h$. By Ball and Govaerts [4], \bar{O} is a minimal blocking set of $PG(6, q)$ of size $q^3 + 1$ which spans the six-dimensional space, and hence, by Theorem 7, choosing \bar{O} as base of the cone K , we can construct the following minimal blocking sets generating the whole space:

- (a) $B' = B_{\bar{O}}$ of size $q^{n+2} + q^{n-1} + 1$ in $PG(r, q^n)$ for $4 \leq r \leq 7$, if $n \geq 3$, in $PG(3, q^n)$ if $n \geq 5$, and in $PG(r, q^n)$ for $5 \leq r \leq 7$, if $n = 2$ (case (a)).
- (b) B' of size at least $q^{n+2} + q^{2n-6} + \varepsilon$ ($\varepsilon = 0$ or 1) in $PG(3, q^n)$ if $n = 3$ or 4 , and B' of size at least $q^4 + 1 + \varepsilon$ ($\varepsilon = 0$ or 1) in $PG(4, q^2)$ (case (c)).
- (c) B' of size $q^4 + q + \varepsilon$ ($\varepsilon = 0$ or 1) in $PG(4, q^2)$, and B' of size $q^6 + q^3 + \varepsilon$ ($\varepsilon = 0$ or 1) in $PG(3, q^4)$ if there exist at least two tangent hyperplanes to \bar{O} at the point Q (case (b)).
- (d) B' of size at least $q^5 + q + \varepsilon$ ($\varepsilon = 0$ or 1) in $PG(3, q^3)$ if there exist at least two tangent hyperplanes to \bar{O} at the point Q (case (d)).

Since the known examples of ovoids of $Q(6, q)$ are the Thas–Kantor ovoids, with $q = 3^h$ and $h \geq 1$, and the Ree–Tits ovoids, with $q = 3^{2h+1}$, $h > 0$, the above constructions produce examples of minimal blocking sets in $PG(r, q^n)$, $3 \leq r \leq 7$, in projective spaces with $q = 3^h$ and $q = 3^{2h+1}$, $h \geq 1$. In particular, if we rewrite case (a) for $n = 2$, we obtain the following theorem.

Theorem 10 *In $PG(r, q)$, for $5 \leq r \leq 7$, $q = 3^{2e}$, $e \geq 1$, there exist minimal blocking sets of size $q^2 + \sqrt{q} + 1$ generating the whole space, and which are the union of Baer sublines through a point.*

Also, we can prove the following result.

Theorem 11 *In $PG(4, q)$, $q = 3^{2e}$, $e \geq 1$, there exist minimal blocking sets of size at least $q^2 + 2$.*

Proof By (6) above, it suffices to prove that it is possible to apply Construction B to an ovoid \bar{O} of $Q(6, q')$, $q' = 3^e$, $h \geq 1$, in such a way that Y is an essential point of $B_{\bar{O}}$. Since \bar{O} is a minimal blocking set of $PG(6, q') = \langle Q(6, q') \rangle$, there exist four-dimensional subspaces of $PG(6, q')$ disjoint from \bar{O} .

Let $q = q'^2$ and consider the Barlotti–Cofman representation $\Pi_4 = \Pi_4(\Sigma', \Sigma, S)$ of $PG(4, q)$ in $\Sigma' = PG(8, q')$.

Let Y and Z be two different elements of the Desarguesian line spread S of $\Sigma = PG(7, q') \subset \Sigma'$, let Ω and T be two distinct points of Y and let Q be a point of Z . Let S_5^* be a five-dimensional S -subspace of Σ containing Y and not containing Z , so $S_5^* \cap \langle Y, Z \rangle = Y$. Let Γ_4' be a four-dimensional subspace of Σ' such that $\Gamma_4' \cap \Sigma = \Gamma_3$ is a three-dimensional subspace of S_5^* disjoint from Y and let Γ_1 be a line contained in $\langle Y, Z \rangle$ passing through Q and disjoint from Y . Note that $S_6 = \langle \Gamma_4', S_5^* \rangle$ defines a hyperplane Π_3 of Π_4 . Also, $\Gamma_6' = \langle \Gamma_4', \Gamma_1 \rangle$ is a six-dimensional subspace of Σ' disjoint from Y containing Q . Let $\Gamma_7' = \langle \Gamma_6', T \rangle$ and embed in Γ_6' an ovoid \bar{O} of a parabolic quadric $Q(6, q')$ in such a way that $\bar{O} \cap \Sigma = \{Q\}$ and $\Gamma_4' \cap \bar{O} = \emptyset$. Applying Construction B to the cone with vertex Ω and base \bar{O} , we obtain an ovoidal blocking set $B_{\bar{O}}$ of Π_4 generating the whole space (see proof of Theorem 7). Since $\Gamma_4' \cap \bar{O} = \emptyset$ and $\Gamma_4' \cap \Sigma = \Gamma_3 \subset S_5^*$, by case (••) of Proposition 8, Y is an essential point of $B_{\bar{O}}$. If there exist at least two tangent hyperplanes to \bar{O} at the point Q , then, by Theorem 7 case (b), $B_{\bar{O}}$ is a minimal blocking set of $PG(4, q)$ of size $q^2 + \sqrt{q} + 1$. Otherwise, by Theorem 7 case (c), $B_{\bar{O}}$ contains a minimal blocking set of $PG(4, q)$ of size at least $q^2 + 2$. \square

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